# ON n-WEAKLY AMENABLE BANACH ALGEBRA[1]

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**ABSTRACT**

It is shown that if a Banach algebra A is a left ideal in its second dual algebra and has a left bounded approximate identity, then the 2-weak amenability of A implies the (2m+2)-weak amenability of A for all m ≥1. In particular, A is 4-weakly amenable.

**Key Words**: Banach algebra, n-weakly amenable, derivation, left ideal, bounded approximate identity

## 1.0. INTRODUCTION

In [5], Dales, Ghahramani and Gronbaek introduced the concept of n-weak amenability for Banach algebras. They determine the relations between m- and n- weak amenability for general Banach algebra and for Banach algebras in various classes. They proved that for n ≥1, (n+2)-weak amenability always implies n-weak amenability. As for the converse, they have raised an open question: Does n-weak amenability implies (n+2)-weak amenability? They also asked whether or not 2-weak amenability implies 4-weak amenability for an arbitrary Banach algebra.

In this note, sufficient conditions under which 2-weak amenability will implies 4-weak amenability, and n-weak amenability will implies (n+2)-weak amenability for even positive integer n is discussed.

## 2.0. PRELIMINARIES

First, we recall some standard notions, some of which are in the text of Bonsall and Duncan [3]. Let A be an algebra, and let X be an A-bimodule with respect to the operations (a,x)'!a.x and (a,x)'!x.a, AxX'!X. Then X is a commutative (symmetric) A-bimodule if ax = xa (a° A, x ° X). A linear map D: A'!X is a derivation if

D(ab) = a.D(b) + D(a).b, (a,b °A).

For any x °X, the mapping $ä_x$: A'!X given by $ä_x$ (a) = a.x - x.a, (a ° A) is a continuous derivation, called an inner derivation.

Let A be a Banach algebra, and let X be an A-bimodule. Then X is a Banach A-bimodule, if X is a Banach space and if there is a constant k such that
$\|a.x\| \leq k\|a\|.\|x\|$ and $\|x.a\|$ $k\|a\|.\|x\|$, $(x \in X, a \in A)$.

Let $B^1(A,X)$ be the space of all continuous derivations from A into X and let $Z^1(A,X)$ be the space of all inner derivations from A into X. Then the first cohomology group of A with coefficients in X is the quotient space $H^1(A,X) = B^1(A,X) / Z^1(A,X)$.

Let A be a Banach algebra, and let X be a Banach A-bimodule. Then $X^*$, the dual space of X is a Banach A-bimodule with respect to the operations
$(a.f)(x) = f(x.a), (f.a)(x) = f(a.x)$, $(a \in A, x \in X, f \in X^*)$,

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**(Footnote)**
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$X^*$ is the dual module of X, and in particular, $A^*$ is the dual module of A. The Banach algebra A is amenable if $H^1(A,X) = \{0\}$ for each Banach A-bimodule X, i.e. if each bounded derivation from A into the dual Banach A-bimodule $X^*$ is inner. A is weakly amenable if $H^1(A,A^*) = \{0\}$.

For each $n \geq 1$, $A^{(n)}$, the nth conjugate space of A, is a Banach A-bimodule, with module actions defined inductively by

$$<u,F.a> = <a.u,F>, \quad <u,a.F> = <u.a,F>, \ (F \in A^{(n)}, u \in A^{(n-1)}, a \in A.$$

A Banach algebra A is called n-weakly amenable $H^1(A,A^{(n)}) = \{0\}$.

Richard Arens defined two products on any dual $A^{**}$ of any Banach algebra A. Each product makes $A^{**}$ into Banach algebra and the canonical injection of k: A'!$A^{**}$ is a homomorphism for both products. As in [8], for m $\geq 1$, we always equip $A^{(2m)}$ with the first Arens product. First Arens product on $A^{**}$ is given by the following formula

$$<f,uv> = <vf,u>, \ (f \in A^*, u,v \in A^{**}) \text{ where } vf \in A^* \text{ is defined by } <a,vf> = <fa,v>, \ (a \in A)$$

## 3.0. MAIN RESULTS

For a Banach space X, we will denote by $X^{(\cdot)}$, the image of X in $X^{(2m)}$ under the canonical mapping. But if no confusion may occur, we keep using X to denote this image. For m > 0, the subspace of $X^{(2m+1)}$ annihilating $X^{(\cdot)}$ will be denoted by X

$X = \{F \in A^{(2m+1)}: F/_{X}^{-} = 0\}$.

Also for a Banach algebra A,

$A = \{F \in A^{(2m+1)}: F/_{A}^{-} = 0\}$

$(A^*) = \{F \in A^{(2m+2)}: F/_{(A^*)}^{-} = 0\}$

and generally,

$(A^{(n)}) = \{F \in A^{(2m+n+1)}: F/(_{A}^{(n)})^{-} = 0\}$.

$(A^{(n)})$ are weak* closed submodule of $A^{(2m+n+1)}$ for all n 1.

The following lemmas are useful in establishing our results

**Lemma 3.1 [8]** Suppose that A is a left, right or two sided ideal in $A^{(**)}$. Then it is also a left, right or two sided ideal in $A^{(2m)}$ for all m 1.

**Remark 3.1** From the prove of the above lemma, Zhang show that if A is a left ideal in $A^{(2m)}$, then it is also a left ideal in $A^{(2m+2)}$. Thus If A is a left ideal in $A^{**}$, then it is also a left ideal of $A^{(2m+n)}$ for even positive integer n.

**Lemma 3.2 [8]** Suppose that A is a Banach algebra with a left (right) bounded approximate identity. Suppose that X is a Banach A-bimodule and Y is a weak* closed submodule of the dual module $X^*$. If the left (respectively right) A-module action on Y is trivial, then

$H^1(A,Y) = \{0\}$.

The following theorems and corollaries are the main results

**Theorem 3.1**

Let A be a 2-weakly amenable Banach algebra. If A has a left (right) bounded approximate identity, and is a left (right) ideal in $A^{**}$, then $\breve{A}$ is (2m+2)-weakly amenable for m 1.

**Proof** We give the prove in the case A has a left b.a.i. and is a left ideal in $A^{**}$. From the A-bimodule direct sum decomposition

$A^{(2m+2)} = \{A^*\} + (A^{**})^{-}$

we have the cohomology group decomposition

$H^1(A,A^{(2m+2)}) = H^1(A,A^{**}) + H^1(A,(A^*)\ )$.
Since A is 2-weakly amenable, then $H^1(A,A^{**}) = \{0\}$, and so
$H^1(A,A^{(2m+2)}) = H^1(A,(A^*))$,
so we need to show that $H^1(A,(A^*)) = \{0\}$. From Lemma 3.1 and the remark that follows, A is a left ideal in $A^{**}$ implies A is also a left ideal in $A^{(2m+2)}$ and so af = 0 for a A, f(A*), thus the left A-module action on (A*) is trivial, and so by Lemma 3.2,
$H^1(A,(A^*)) = \{0\}$.Thus, $H^1(A,A^{(2m+2)}) = \{0\}$. That is, A is (2m+2)-weakly amenable.

**Corollary 3.1** Let A be a 2-weakly amenable Banach algebra. If A has a left (right) bounded approximate identity, and is a left (right) ideal in $A^{**}$, then A is 4-weakly amenable.

**Proof** : This is a case of m=1 in the above Theorem 3.1

**Remark 3.2** Corollary 3.1 gives sufficient conditions under which 2-weak amenability will imply 4-weak amenability for a general Banach algebra, and thus answer one of the questions raised by the authors in [5].

From the A-bimodule direct sum decompositions
$A^{(2m+1)} = (A) + (A^*)^{\cdot}$
$A^{(2m+2)} = (A^*) + (A^{**})^{\cdot}$
$A^{(2m+3)} = (A^{**}) + (A^{***})^{\cdot}$
We have in general,
$A^{(2m+n)} = (A^{(n-1)}) + (A^{(n)})^{\cdot}$
for all n 1.

We prove the next result.

**Theorem 3.2** Let A be an n-weakly amenable Banach algebra for some even positive integer n. Suppose A has a left (right) bounded approximate identity, and is a left (right) ideal in $A^{**}$, then A (2m+n)-weakly amenable for m1.

**Proof** We give the prove in the case A has a left b.a.i. and is a left ideal in A**.
From the general A-bimodule direct sum decomposition,
$A^{(2m+n)} = \{A^{(n-1)}\} + (A^{(n)})^{\cdot}$
We have the cohomology group decomposition,
$H^1(A,A^{(2m+n)}) = H^1(A,A^{(n)}) + H^1(A,(A^{(n-1)}))$
since A is n-weakly amenable, then $H^1(A,A^{(n)}) = \{0\}$, and so
$H^1(A,A^{(2m+n)}) = H^1(A,(A^{(n-1)}))$,
so we need to show that $H^1(A,(A^{(n-1)})) = \{0\}$, where,
$(A^{(n-1)}) = \{F \in A^{(2m+n)} : F/(_A{}^{(n-1)})^{-} = 0\}$.
From Lemma 3.1 and the remark that follows, A is a left ideal in $A^{**}$ implies A is also a left ideal in $A^{(2m+n)}$ for all even positive integer n and so af = 0 for a A, $f(A^{(n-1)}\}$, thus the left A-module action on $(A^{(n-1)})$ is trivial, and so by Lemma 3.2,
$H^1(A,(A^{(n)})) = \{0\}$.Thus, $H^1(A,A^{(2m+n)}) = \{0\}$. That is, A is (2m+n)-weakly amenable.

**Corollary 3.2** Let A be a n-weakly amenable Banach algebra for some even positive integer n. Suppose A has a left (right) bounded approximate identity, and is a left (right) ideal in $A^{**}$, then A is (n+2)-weakly amenable for m 1.

**Proof** This is a case of m=1 in the above Theorem3.3.

**Remark 3.3**

Corollary3.2 is the prove of the partial converse to [5, Propostion1.2] for case where n is even positive integer.

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