THE *N*-SQUARE PROBLEM

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Abstract

In this paper, we consider the historical problem of finding integers n satisfying the so-called N-Square Law, viz:

$$\left(\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2\right) = \sum_{i=1}^{n} z_i^2$$

where the z_i are bilinear functions of x's and y's. this law has been established for integers n = 1, 2, 4, and 8 only. However, from our study of Bol quaternion algebras we have a strong conjecture that it holds true for higher orders. Considering that this result holds for alternative algebras, we investigate the case for non-alternative algebras and obtained suitable conditions for possible extension of the result. Our result shows that the N-square law may exist for $n \geq 16$.

Keywords: N-squares, quaternions, Bol, norm, alternative, composition, algebras.

Introduction

This paper considers loop algebras whose underlying loops are non-associative Bol loops of order 2^k where $k \geq 3$ is an integer.

These are Bol algebras, which in the language of Glauberman and Wright (1968) might be called Bol–2 algebras (respectively Bol–2 loops).

We shall however call them Bol-quaternion algebras since they exhibit quaternion like behaviour. Bol quaternion algebras are of wide applications in many scientific research e.g. computer science, geometry, mathematical physics, mechanics, etc.

Definition 1

Let A be a loop algebra over a real field F whose basis elements generate a non-abelian Bol loop L of order n, where:

$L = \{<e_i>: e_0 = 1, e_i^2 = -1 \text{ and } e_ie_j = \pm e_je_i \text{ for } 1 \leq i \leq n-1\}$ ..... (1)

Then A is said to be a Bol algebra of quaternion type and L is called a Bol loop of quaternion type (or simply Bol quaternion loop).

An element x of a Bol loop L such that $x^2 = -1$, $x^4 = 1$ and $xy = \pm yx$ for all y in L is called a Bol quaternion element.

Definition 2

A loop (L,.) satisfying the property that $(xy \,.\, z)\, y = x\, (yz \,.\, y)$ for all x, y, z, in L is called a Right Bol loop. A loop (L,) is called a Left Bol loop if for all x, y, z, in L. $y\,(z.\, yx) = (y.\, zy)\, x$.

These identities are duals of each other, and a loop satisfying both identities is called a Moufang loop. A loop satisfying either of the identities is called a Bol loop. Bol and Moufang loops are well known in the literature, see

Burn (1978), Chein (1999), Kiechle H and Kinyon M. K. (2004), Solarin and Sharma (1987), etc. Burn (1978) showed that there are exactly 6 non-associative Bol loops of order 8.

Only one of these (named II1 in the paper) is of quaternion type. We shall name it BQ_8 in accordance with the numenclature of its associative counterpart (the quaternion group Q_8) so that

$$BQ_8 = \{ < i, j, k >: ij = ji, jk = kj, ik = - ki \text{ and } i^2 = j^2 = k^2 = -1 \}$$

The construction and classification of Bol loops of order 16 were undertaken by Solarin and Sharma (1987), and Moor house (2001) among others. Moor house showed that there are only 2, 038 non-associative Bol loops of order 16, and 37 of these have 1 involution and non-trivial centers. He listed these 37 loops in isotopy classes 0 to 35 with only one of them, the loop $M_{16}\, Q_8$ (which he coded 16;1.2.31), being Moufang.

Since these loops are of the type defined by (1) above, we shall name them BQ_{16} (1), BQ_{16} (2), BQ_{16} (37). Interested readers may obtain the respective Cayley tables from the site http: ||Math.Uwyo.Edu|Moorhous|Pub| bol16.html.

For loops of higher orders, there are no known examples of Bol loops of type (1) of order 32 or higher (and there may not be).

Efforts in this regard only yield some quaternion like loops which are not Bol; see Q_{32} (1) (Table 1) for instance, or the sedenions tagged "interesting loops" by Cawagas (see Naggy and Vojtechovisky, 2005).

Materials and method

A popular problem in Mathematics which has engaged the attention of many researchers is the problem of finding integers n satisfying the norm identity.

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{j=1}^{n} y_j^2\right) = \sum_{k=1}^{n} z_k^2 \qquad \text{..... (2)}$$

where z_k are bilinear functions of x_i and y_j.

This problem called the N-square problem was solved by Adolf Hurwitz who showed that if F is of char $\neq 2$ then (2) has solution if and only if n =1, 2, 4, 8. The Hurwtz theorem which arose principally from the fact that it is only in these dimensions that we have finite dimensional real division algebras (Bott and Milnor, 1958) did not deter researchers from considering possible extensions to n since the algebras considered are alternative algebras.

Dickson E. (1906), for instance, devised the "doubling process" which takes us from n = 1 (Real) to n = 2 (Complex) to n = 4 (Quaternions) and to n = 8 (Cayley algebras) and then to ... because the next algebra is not a division algebra. Slinko (2002) considered the extension of n to higher orders by investigating some Lie algebras, etc. Yet no new result extending n has been obtained. Considering that the algebras R, C, Q and O in the Hurwitz theorem are alternative Bol quaternion algebras, we take a look at the structure of Bol quaternion algebras, in particular the non-alternative ones for the extension of n, if at all.

Definition 3

An algebra A over F is said to be alternative if for all x, y, in A the identities

$$x^2 y = x (xy) \text{ and } yx^2 = (yx) x$$

known respectively as the Left and Right Alternative laws, hold in A. An algebra A is called Left (respectively Right) alternative algebra if only the Left (or, the Right) alternative law holds in A.

Definition 4

An involution of an algebra A is a linear map $q \longrightarrow \overline{q}$ of A satisfying.

$\overline{q_1} . \overline{q_2} = \overline{q_2} . \overline{q_1}$ and $q \overline{\overline{=}} q$ for all q_1, q_2, q_3 in A.

For any q in A, q $\neq$ 0, the products $q . \overline{q} = N(q)$ and (N(q))-1. q are called the norm of q and the inverse of q respectively. If 1 is the identity of A then $q . q^{-1} = q^{-1} . q = 1$

An algebra A satisfying the equation (2) above is called a composition algebra or normed algebra. This is because such algebras are equipped with a non degenerate quadratic form N such that for every q_1, q_2, in A

$$N(q_1) . N(q_2) = N(q_1 . q_2) \qquad \text{..... (3)}$$

which is an equivalent expression for (2) in terms of the norm *N*.

It is not difficult to see that the quaternion algebra Q (defined over Q_8) provides this solution for $n = 4$ since for any q in Q.

$$N(q) = (a_0 + \sum_{i=1}^{3} a_i e_i)(a_0 - \sum_{i=1}^{3} a_i e_i) = \sum_{i=0}^{3} a_i^2$$

Thus, for all $q_1 = a_0 + \sum_{i=1} a_i e_i$ and $q_2 = b_0 + \sum_{j=1} b_j e_j$ in Q, we have, Goodaire *et al.* (1999), $N(q_1). N(q_2) = (a_0b_0 - \Sigma^3 a_i b_i)^2 + (a_0b_i + a_1b_0 + a_2b_3 - a_3b_2)^2 + (a_0b_2 + a_2b_0 \; a_1b_3 + a_3b_1)_2 + (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)2 = N(q_1 q_2)$

Similarly, the Octonium algebra (defined over the Bol Loop $M_{16}Q_8$) provides the solution for $n = 8$. Since these are alternative divisions algebras, the norm of any element q is a scalar $\sum_{i=1}^{n-1} a_i^2$. Hence (3) holds true over *F* of characteristic *p* (*p* prime) but for the non-alternative cases the norm is not a scalar. The following theorem states the argument for the Bol quaternion algebra defined over the Bol loop BQ_8 since in this case the norm of any *q* in A is not a scalar.

Main results

Theorem 1

Let A be the algebra generated by the Bol loop BQ_8 then A is a composition algebra if and only if char(F) is 2.

Proof

To prove that A is a composition algebra, we must show that for all $q_1, q_2 \in$ A the norm equation $N(q_1) N(q_2) = N(q_1q_2)$ holds true.

From the definition of the norm, we have for any q_1, q_2 in A

$$N(q_1) = \sum_{i=0}^{3} a_i^2 + 2[a_2a_3e_1 - a_1a_2e_3] \text{ and } N(q_2) = \sum_{j=0}^{3} b_j^2 + 2[b_2b_3e_1 - b_1b_2e_3]$$

Hence, $N(q_1) N(q_2) = (\Sigma^3_{i=0} a_i^2)(\Sigma^3_{j=0} b_j^2) + 2(\Sigma^3_{i=0} a_i^2) b_2b_3e_1 - 2(\Sigma^3_{i=0} a_i^2) b_1b_2e_3 + 2(\Sigma^3_{j=0} b_j^2) a_2a_3e_1 - 4a_2a_3b_2b_3 + 4a_2a_3b_1b_2e_1 - 2a_1a_2(\Sigma^3_{j=0} b_j^2)e_3 - 4a_1a_2b_2b_3e_2 - 4a_1a_2b_1b_2 = \Sigma^3_{i,j=0}(a_i b_j)^2 - 4a_2b_2[a_3b_3 + a_1b_1] + 2[(\Sigma^3_{i=0} a_i^2)b_2b_3 + a_2a_3(\Sigma^3_{j=0} b_j^2)e_1 +$

$4a_2b_2[a_3b_1 - a_1b_3]e_2 - 2[(\Sigma^3_{i=0} a_i^2)b_1b_2 + a_1a_2(\Sigma^3_{j=0} b_j^2)e_3 = \Sigma^3_{i,j=0}(a_i b_j)^2$

only if $a_2b_2 = 0,\ a_3b_3 = -a_1b_1,\ a_2a_3 = 0,\ a_1b_3 = a_3b_1,$

$$a_1a_2 = 0,\ b_1b_2 = 0, = \sum_{i=0}^{3} a_i^2 = 0, \quad \text{and} \quad \sum_{j=0}^{3} b_j^2 = 0 \qquad \text{..... (4)}$$

or if char(F) = 2 Now, for any q_1, q_2 in A we have:

$$q_1q_2 = a_0b_0 - \sum_{i=1}^{3} a_i b_i + (a_0b_1 + a_1b_0 - a_2b_3 - a_3b_2)e_1$$
$$+ (a_0b_2 + a_2b_0 + a_1b_3 - a_3b_1)e_2 + (a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1)e_3$$

Hence, by definition of the norm

$N(q_1q_2) = (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)2 + (a_0b_1 + a_1b_0 - a_2b_3 - a_3b_2)2 + (a_0b_2 + a_2b_0 + a_1b_3 - a_3b_1)2 + (a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1)2 + 2(a_0b_2 + a_2b_0 + a_1b_3 - a_3b_1)(a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1)e_1 - 2(a_0b_1 + a_1b_0 - a_2b_3 - a_3b_2)(a_0b_2 + a_2b_0 + a_1b_3 - a_3b_1)e_3$

On expanding these brackets and simplifying we obtain

$N(q_1q_2) = (\Sigma^3_{i=0} (a_ib_i)^2\ 2a_0b_0(\Sigma^3_{i=1} a_ib_i) + 2\Sigma_{1<i<j<3} a_ib_ia_jb_j + \Sigma^3_{i\neq j=0} (a_ib_j)^2 + 2\Sigma_{0<i<j<3} a_ib_ja_jb_i - 2a_0b_1a_3b_2 + 2a_0b_2 (a_1b_3 - a_3b_1) + 2a_0b_3a_1b_2 + 2[a_0b_2(a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1) + a_2b_0(a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1) + a_1b_3(a_0b_3 + a_3b_0 + a_1b_0 + a_1b_2 + a_2b_1) - a_3b_1 (a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1)]e_1 - 2[a_0b_1(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + a_1b_0(a_0b_2 + a_2b_0 + a_1b_3 - a_3b_1) + a_2b_3(a_0b_2 + a_2b_0 + a_1b_3 - a_3b_1) - a_3b_2 (a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1)]e_3$

$= \Sigma^3_{i=j=0}(a_ib_i)^2 + \Sigma^3_{i\neq j=0} (a_ib_j)^2 + 4a_2b_2 (a_1b_1 - a_3b_3) + 4a_0b_2(a_1b_3 - a_3b_1) + 2[a_0b_2(a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1) + a_2b_0(a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1) + a_1b_3(a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1) - a_3b_1 (a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1)]e_1 - 2[a_0b_1(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1) + a_1b_0(a_0b_2 + a_2b_0 + a_1b_3 - a_3b_1) - a_2b_3(a_0b_2 + a_2b_0 + a_1b_3 - a_3b_1) - a_3b_2 (a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1)]e_3$

$= \Sigma^3_{i=j=0}(a_ib_i)^2 + \Sigma^3_{i\neq j=0} (a_ib_j)^2$ only if

$a_1b_1 = - a_3b_3,\ a_2b_2 = 0,\ a_0b_2 = 0,\ a_0b_2 = - a_2b_0,\ a_1b_3 = a_3b_1,\ a_0b_3 = - a_3b_0,$
$a_1b_2 = - a_2b_1,\ a_0b_1 = - a_1b_0$ and $a_2b_3 = - a_3b_2$ ..... (5)

or if char (F) = 2 Since equations (4) and (5) can only be satisfied if A is the zero algebra, or the char(F) is 2, this establishes the second part of theorem.

For the first part of the theorem, we must recognize that if char (F) = 2 then

$N(q_1) = \Sigma^3_{i=0} a_i^2$ and $N(q_2) = \Sigma^3_{j=0} b_j^2$ so that $N(q_1)\,N(q_2) = (\sum_{i=0}^{3} a_i^2)(\sum_{j=0}^{3} b_j^2)$

which on expansion yields:

$N(q_1)\ N(q_2) = (a_0b_0)^2 + (a_1b_1)^2 + (a_2b_2)^2 + (a_3b_3)^2)^2 + (a_0b_1)^2 + (a_1b_0)^2 + (a_2b_3)^2 + (a_3b_2)^2 + (a_0b_2)^2 + (a_2b_0)^2 + (a_1b_3)^2 +(a_3b_1)^2 + (a_0b_3 + a_3b_0 + a_1b_2 + (a_2b_1)^2$

Similarly, if char(F) is 2, then from the definition of the norm of q_1q_2 we have

$N(q_1q_2) = (a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3)^2 + (a_0b_1 + a_1b_0 - a_2b_3 - a_3b_2)^2 + (a_0b_2 + a_2b_0 + a_1b_3 - a_3b_1)^2 + (a_0b_3 + a_3b_0 + a_1b_2 + a_2b_1)^2$ .....(6)

which on expansion and simplification yields:

$N(q_1q_2) = (a_0b_0)^2 + (a_1b_1)^2 + (a_2b_2)^2 + (a_3b_3)^2)^2 + (a_0b_1)^2 + (a_1b_0)^2 + (a_2b_3)^2 + (a_3b_2)^2 + (a_0b_2)^2 + (a_2b_0)^2 + (a_1b_3)^2 +(a_3b_1)^2 + (a_0b_3 + a_3b_0 + a_1b_2 + (a_2b_1)^2$

Consequently, $N(q_1q_2) = \Sigma 3_{i=0}(a_ib_i)^2 + \Sigma^3_{i\neq j=0} (a_ib_j)^2 = \Sigma^3_{i,j=0}(a_ib_j)^2 = N(q_1)\ N(q_2)$

Remarks: By the same argument, as above, we investigated the algebras defined over the non-Moufang Bol Loops BQ (i) and obtained the following corollary. The next theorem summarizes it all.

Corollary: Let A_i be the loop algebras generated by the non-Moufang loops $BQ_{16}(i)$ over the real field F then A_i is a composition algebra if and only if char(F) = 2.

Theorem 2

Let A be an algebra generated by a Bol quaternion loop L over the real field F, then A is a composition algebra if and only if Char(F) is 2 or $xy = y^{-1}x$ for all $x, y \in$ L where $x, y \notin \{1, -1\}$.

Proof

Let A be a composition algebra, then for all $q_1, q_2 \in$ A we must have:

$$N(q_1) = \sum_{i=0}^{n-1} a_i^2 - 2 \sum_{1<i<j}^{n-1} a_ia_je_ie_j = \sum_{i=0}^{n-1} a_i^2$$

which implies that $2\ \Sigma^{n-1}_{1<i<j} a_ia_je_ie_j = 0$. This holds only if char(F) = 2 or if $e_i e_j = -e_je_i = e_j^{-1} e_i$ for all $e_i, e_j \in$ L, $i \neq 0, j \neq 0$. Similarly, $N(q_2) = \Sigma^{n-1}_{j=0} b_j^2$ so that

$$N(q_1)\ N(q_2) = (\sum_{i=0}^{n-1} a_i^2)(\sum_{j=0}^{n-1} b_j^2) = \sum_{i,j=0}^{n-1} (a_ib_j)^2$$

$$N(q_1q_2) = (a_0b_0 - \sum_{i=1}^{n-1} a_i b_i)^2 (\sum_{k=1}^{n-1} \sum_{i\neq j=0}^{n-1} a_ib_je_k)^2$$

Since:

$$(\sum_{k=1}^{n-1} \sum_{i\neq j=0}^{n-1} a_ib_je_k)^2 = [(a_0b_1 + a_1b_0 + \sum_{0<i\neq j\neq 1} a_ib_j)e_1 + (a_0b_2 + a_2b_0 + \sum_{0<i\neq j\neq 2} a_ib_j)e_2 +$$

$$+, \dots, + (a_0b_{n-1} + a_{n-1} b_0 + \sum_{0<i\neq j\neq n-1} a_ib_j)e_{n-1}]^2$$

$$= -(a_0b_1 + a_1b_0 + \sum_{e_1,\, 0<i\neq j\neq 1} a_ib_j)^2 e_1 + (a_0b_2 + a_2b_0 + \sum_{e_2,\, 0<i\neq j\neq 2} a_ib_j)^2 +$$

$$- \dots - (a_0b_{n-1} + a_{n-1} b_0 + \sum_{e_{n-1},\, 0<i\neq j\neq n-1} a_ib_j)^2$$

$$+ 2 \sum_{1<r<s<n-1} (\sum_{e_r,\, i\neq j} a_ib_j) (\sum_{e_s,\, k\neq l} a_kb_l)e_r.e_s \qquad \dots\dots(7)$$

where $e_r.e_s = e_s.e_r$, we have:

$$N(q_1q_2) = (a_0b_0 - \sum_{i=1}^{n-1} a_i b_i)^2 + (\sum_{k=1}^{n-1} (a_0b_k + a_kb_0 + \sum_{e_k,\, 0<i\neq j\neq k} a_ib_j)^2$$

$$-2 \sum_{1<r<s<n-1} (\sum_{e_r,\, i\neq j} a_ib_j) (\sum_{e_s,\, k\neq l} a_kb_l)e_r.e_s \qquad \dots\dots(8)$$

where $e_r.e_s = e_s.e_r$ Thus, if A is a composition algebra, then:

$$N(q_1q_2) = (a_0b_0 - \sum_{i=1}^{n-1} a_i b_i)^2 + (\sum_{k=1}^{n-1} (a_0b_k + a_kb_0 + \sum_{e_k,\, 0<i\neq j\neq k} a_ib_j)^2 \qquad \dots\dots (9)$$

which is the required sum of N-squared terms. But this implies that:

$$-2 \sum_{1<r<s<n-1} (\sum_{e_r,\, i\neq j} a_ib_j) (\sum_{e_s,\, k\neq l} a_kb_l)e_r . e_s = 0$$

which is true only when char(F) = 2 or if $e_re_s = - e_se_r = e_s{-}1e_r$ for all non-central elements e_r, e_s. To prove the converse, let L be such that $xy = y^{-1} x$ for all $x, y \in$ L where $x, y \notin \{1, -1\}$ then (8) reduces to (9). And since

$$(a_0b_0 - \sum_{i=1}^{n-1} a_i b_i)^2 = (a_0b_0)^2 + (\sum_{i=1}^{n-1} a_ib_i)^2 - 2a_0b_0 \sum_{i=1}^{n-1} a_ib_i$$

we have on simplifying further

$$(a_0b_0 - \sum_{i=1}^{n-1} a_i b_i)^2 = \sum_{i=1}^{n-1} (a_ib_i)^2 - 2a_0b_0 \sum_{i=1}^{n-1} a_ib_i + 2 \sum_{1<i<j<n-1} a_ib_ia_jb_j \qquad \dots\dots (10)$$

Therefore,

$$N(q_1q_2) = \sum_{i=1}^{n-1} (a_i b_i)^2 - 2a_0b_0 \sum_{i=1}^{n-1} a_ib_i + 2 \sum_{1<i<j<n-1} a_ib_i\, a_jb_j + (a_0b_k + a_kb_0 + \sum_{e_k,\, 0<i\neq j\neq k} a_ib_j)^2$$

Also,

$$(a_0b_k + a_kb_0 + \sum_{e_k,\, 0<i\neq j\neq k}^{n-1} a_ib_j)^2 = (\sum_{i\neq j=0} (a_i b_i)^2 + 2a_0b_1a_1b_0 \pm 2 \sum_{i\neq j\neq 0} a_ib_ja_jb_i +$$

$$2 \sum_{i\neq j\neq k\neq l} a_ib_ja_kb_l)e_i + (\sum_{i\neq j=0}^{n-1} (a_i b_i)^2 + 2a_0b_2a_2b_0 \pm 2 \sum_{i\neq j\neq 0} a_ib_ja_jb_i + 2 \sum_{i\neq j\neq k\neq l} a_ib_ja_kb_l)e_2$$

$$+ \dots + (\sum_{i\neq j=0}^{n-1} (a_i b_i)^2 + 2a_0b_{n-1} a_{n-1}b_0 \pm 2 \sum_{i\neq j\neq 0} a_ib_ja_jb_i + 2 \sum_{i\neq j\neq k\neq l} a_ib_ja_kb_l)e_{n-1}$$

$= 2a_0b_0 (\sum_{i=1}^{n-1} a_i b_i + \sum_{e_\alpha} \sum_{i\neq j} (a_i b_i)^2 + \pm 2 \sum_{i\neq j\neq 0} a_ib_ja_jb_i + 2 \sum_{i\neq j\neq k\neq l} a_ib_ja_kb_l)$

Hence,

$N(q_1q_2) = \sum_{i=0}^{n-1} (a_i b_i)^2 + 2 \sum_{1<i<j<n-1} a_ib_ia_jb_j + \sum_{e_\alpha} (\sum_{i\neq j}^{n-1} (a_ib_j)^2)$

$= \pm 2 \sum_{e_\alpha} (\sum_{i\neq j\neq 0} a_ib_ia_jb_j) \pm 2 \sum_{e_\alpha} (\sum_{i\neq j\neq k\neq l} a_ib_ja_kb_l)$

$= \sum_{i,j=0}^{n-1} (a_i b_i)^2) + 2 \sum_{1<i<j<n-1} a_ib_ia_jb_j \pm 2 \sum_{e_\alpha} (\sum_{i\neq j\neq 0} a_ib_ia_jb_j) \pm 2 \sum_{e_\alpha} (\sum_{i\neq j\neq k\neq l} a_ib_ja_kb_l)$

giving $n^2 + n!$ terms in the expansion. Since the n! terms automatically cancel out as a result of the Bol quaternion structure of A, we are left with the n^2 terms. That is:

$N(q_1q_2) = \sum^{n-1}_{i,j=0} (a_i b_i)^2)$

Similarly, let char(F) = 2, then (8) reduces easily to (9) which on expansion becomes $N(q_1q_2) = \sum^{n-1}_{i,j=0} (a_i b_i)^2$

Now, consider that if q_1, q_2 are in A, we have:

$N(q_1) = \sum_{i=0}^{n-1} a_i^2 - 2 \sum_{1<i<j}^{n-1} a_ia_je_ie_j$ and $N(q_2) = \sum_{i=0}^{n-1} b_j^2 \; 2 \sum_{1<i<j}^{n-1} b_ib_je_ie_j$

where e_i, e_j are commuting pairs. Thus, if L is such that $e_i.e_j = -e_je_i = e_j^{-1}e_i$ for all $e_i, e_j \notin \{-1, 1\}$, or char(F) = 2 then $(Nq_1) = \sum^{n-1}_{i=0} a_i^2$ and $(Nq_2) = \sum^{n-1}_{j=0} b_j^2$. Hence, $(Nq_1)(Nq_2) = (\sum^{n-1}_{i=0} a_i^2)(\sum^{n-1}_{j=0} b_j^2) = \sum^{n-1}_{i,j=0} (a_i b_i)^2 = N(q_1q_2)$

Therefore, (3) is established for all Bol quaternion algebras satisfying $xy = y^{-1}x$ where $x, y \notin \{1, -1\}$, or, for which char(F) is 2.

This establishes the theorem.

Discussion

As shown in the proof of theorem (1), it is not possible obtaining a non-zero non-alternative Bol quaternion algebra whose underlying field is not of characteristics 2 satisfying the set of equations as in (4) and (5). The next theorem (Theorem 2) emphasizes the implications of this for non-alternative Bol quaternion algebras of dimensions higher than 8. The concluding theorem below shows that the condition $xy = y^{-1}x$ for $x, y \notin \{1, -1\}$, though necessary for A to be a composition algebra, is not sufficient unless A has the Bol quaternion structure. It shows that if A is a quaternion algebra satisfying this condition with char(F) ≠ 2 but lacks the Bol identity then A cannot be a composition algebra. Thus, for an algebra A(L) of order $n > 8$ with char(F) ≠ 2 to satisfy the N-square law, A must have the Bol quaternion structure and satisfy $xy = y^{-1}x$ for $x, y \notin \{1, -1\}$ where $x, y \in$ L.

It is of course interesting to note that the non-existence of a Moufang loop of quaternion type for orders greater than 16 is responsible for the non-existence of the N-Square law for orders greater than 8.

Theorem 3

Let A(L) be the loop algebra generated by the loop L = $Q_{32}(1)$ whose table is as given below over a field F of characteristic different from 2 then:

(i) $xy = y^{-1}x$ for all x, y in L where $x, y \notin \{1, -1\}$.
(ii) A(L) is not a composition algebra.

Proof

(i) is obvious. To prove (ii), let $q_1, q_2 \in$ A(L), then by definition:

$N(q_1) = \sum_{i=0}^{15} a_i^2 - 2 \sum^{15} a_ia_je_ie_j = \sum_{i=0}^{15} a_i^2$

since $e_i.e_j = -e_je_i = e_j^{-1}e_i$ for all $e_i, e_j \in L$. Similarly, $(Nq_2) = \Sigma^{15}_{j=0} b_j^2$. Hence,

$$N(q_1)\,(Nq_2) = (\sum_{i=0}^{15} a_i^2\,(\sum_{j=0}^{15} b_j^2) = \sum_{i,j=0}^{15} (a_i\, b_j)^2$$

Now, since $e_i.e_j = -e_je_i$ for all non-central elements e_i, e_j we have by (2.29):

$$N(q_1q_2) = (a_0b_0 - \sum_{i=1}^{15} a_i\, b_i\,)^2 + (\sum_{k=1}^{15} (a_0b_k + a_kb_0 + \sum_{e_k,\ 0<i\neq j\neq k} (a_ib_j\,))^2$$

$$= \sum_{i,j=0}^{15} (a_i\, b_j\,)^2 \pm 4 \sum_{0<i\neq j\neq k}^{15} (a_ib_ja_kb_l) \neq N(q_1)\,(Nq_2)$$

Hence, A(L) is not a composition algebra.

Table 1: Quaternion Loop of order 32 ($Q_{32}(1)$).

*	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_1	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$	$-e_9$	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	e_{14}
e_2	e_2	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5	$-e_{10}$	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_3	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4	$-e_{11}$	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_4	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$	$-e_{12}$	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_5	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$	$-e_{13}$	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$
e_6	e_6	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1	$-e_{14}$	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	$-e_9$
e_7	e_7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1	$-e_{15}$	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	-1	e_1	$-e_2$	$-e_3$	e_4	e_5	$-e_6$	$-e_7$
e_9	e_9	$-e_8$	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	e_3	$-e_2$	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	e_2	$-e_3$	-1	$-e_1$	e_6	$-e_7$	e_4	$-e_5$
e_{11}	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	e_3	e_2	e_1	-1	$-e_7$	$-e_6$	$-e_5$	$-e_4$
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	$-e_6$	e_7	-1	$-e_1$	e_2	$-e_3$
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	e_6	e_1	-1	$-e_3$	$-e_2$
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9	e_6	$-e_7$	$-e_4$	e_5	$-e_2$	e_3	-1	e_1
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$	e_7	e_6	e_5	e_4	e_3	e_2	e_1	-1

Cayley Table of one of the Quaternion Loops of order 32 obtained.

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