ON ORDER STATISTICS FROM WEIBULL-EXPONENTIAL DISTRIBUTION

Olosunde, A.A.
Department of Statistics, University of Agriculture
P.M.B. 2240, Abeokuta
e-mail: akinolosunde@yahoo.ca

Abstract

In this paper we presented an overview portion of distribution theory which is currently under intense developments. The starting point of this topic is the introduction of the distribution which belongs to burr type xii family of distributions called weibull-exponential distribution, the connected area is becoming increasingly broad, and its extension to order statistics made the field so important. Some of the properties of its order statistics which includes; recurrence relations for negative and fractional moments of single order statistics are obtained. The final part of this paper discussed the product moments of two order statistics drawn from weibull-exponential distribution and their applications.

Keywords: order statistics; weibull-exponential; recurrence relations; negative and fractional moment.

Introduction

Weibull-Exponential was first studied by Shah and Dave (1963). The probability density function was given by

$$f(x)=\frac{\delta e^{\gamma}x^{\delta-1}}{(1+e^{\gamma}x^{\delta})^{2}}, \qquad x\geq 0,\delta>0 \qquad \ldots\ldots 1.1$$

where δ and e^{γ} are shape and scale parameters respectively. The distribution is special case of Burr type XII family of distributions (Tadikamalla, 1980), The distribution has played an important role in statistical modeling for instance Dubey (1966) fitted them to business failure data. The density in 1.1 is unimodal; when $\delta \geq 1$, the mode is at $x = 0$ (giving a reverse *J*-shaped curve), and when $\delta < 1$, the mode is at

$$x=e^{-\gamma}\frac{(\delta-1)}{(\delta+1)} \qquad \ldots\ldots 1.2$$

The cumulative distribution function is

$$F(x)=\frac{1}{1+e^{-\gamma}x^{-\delta}}, \qquad x\geq 0,\delta>0 \qquad \ldots\ldots 1.3$$

and the rth raw moment of X is given by

$$E[X^{r}]=e^{-r\gamma/\delta}\frac{r\pi}{\delta}\operatorname{cosec}\frac{r\pi}{\delta}, \qquad r=1,2,\ldots\ldots\ldots \qquad \ldots\ldots 1.4$$

Johnson and Tadikamalla (1982) have discussed methods of fitting the four-parameter form of the Weibull-Exponential distributions. Shoukri *et al* (1988) examined the probability-weighted moment estimators for the three-parameter form of the distributions and compared them to the maximum likelihood estimators.

Distribution of an order statistic

Order statistics have played significant role especially in robust location estimations, detection of outliers, censored sampling, quality control, characterizations and goodness of fit test, just to mention a few. In this paper we derived order statistics from this distribution and establish some recurrence relations satisfied by the single and product moments of order statistics from Weibull-Exponential distributions and give an illustration of its area of application.

Suppose that X_1..........X_n are n jointly distributed random variables. The corresponding order statistics are X_i's arranged in nondecreasing order. The smallest of the X_i's is denoted by $X_{1:n}$, the second smallest is denoted by $X_{2:n}$,......., and, finally, the largest is denoted by $X_{n:n}$. Thus $X_1, X_2, \ldots, X_n$ which in the context we assume is a random sample from an absolutely continuous population with probability density function as in 1.1 above, and cumulative distribution function as in 1.3; let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ be the order statistics obtained by arranging the preceding random sample in an increasing order of magnitude. Then, from 1.1 and 1.3 above the probability density function and cumulative density function of order statistics of Weibull-Exponential distribution is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} u^{i-1}(1-u)^{n-i} \quad , -\infty < x < \infty \; (2.1) <$$

$$\int_0^{F(x)} \frac{n!}{(i-1)!(n-i)!} t^{i-1}(1-t)^{n-i}\, dt , \qquad -\infty < x < \infty$$

..... (2.2)

by noting that

$$F(x)(1-F(x)) = \frac{x}{\delta} f(x) \qquad .....(2.3)$$

and

$$(1-F(x))^2 = \frac{x^{1-\delta}}{\delta e^{\gamma}} f(x) \qquad .....(2.4)$$

where the distribution function can easily be expressed as:

$$F(x) = 1 - \frac{1}{(1+e^{\gamma}x^{\delta})}, \qquad 0 \le x < \infty$$

let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the order statistics obtained from Weibull-Exponential distribution. Let us denote

$$\alpha_{i:n}^{(m)} = E(U_{i:n}^{m}) = \int_0^1 u^m f_{i:n}(u)du, \qquad 1 \le i \le n \qquad ..... (2.5)$$

and

$$\alpha_{i,j:n}^{(m_i,m_j)} = E(U_{i:n}^{m_i} U_{j:n}^{m_j}) = \int_0^1 \int_0^{u_j} u_i^{m_i} u_j^{m_j} f_{i,j:n}(u_i,u_j)du_i du_j$$

then ..... (2.6)

$$\alpha_{i:n}^{(m_i)} = C_{i:n} \int_0^{\infty} u^{m_i}[F(u)]^{i-1}[1-F(u)]^{n-i} f(u)du$$

..... (2.7)

(2.5) and (2.6) are mth moments of single and joint two order statistics respectively and

$$\alpha_{i,j:n}^{(l,k)} = C_{i,j:n} \int_0^{\infty} \int_x^{\infty} x^l y^k [F(x)]^{r-1}[F(y)-F(x)]^{i-j-1}[1-F(y)]^{n-j} * f(x)f(y)dydx \qquad ..... (2.8)$$

Where:

$$C_{i:n} = \frac{n!}{(i-1)!(n-i)!}$$

and

$$C_{i,j:n} = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

Ragab and Green (1984) have obtained recurrence relations for $\alpha_{i:n}^{(k)}$ and an expression for $\alpha_{i,j:n}$ and have derived the distribution of $\frac{X_{i:n}}{X_{j:n}}$. In this paper, I have obtained recurrence relations for $\alpha_{i:n}^{(k)}, \alpha_{i:n}^{(k-p)}, \alpha_{i,j:n}^{(l,k)}$, *and* $\alpha_{i,j:n}^{(l,k-p)}$ utilizing the results developed by Khan *et al.* (1983a and 1983b). It may be noted that for δ>1, k-δ may be negative. Thus adjusting k, we can find inverse and fractional moment of order statistics and moment of ratio of two order statistics. Also, a relation between negative and positive moment of all order statistics is given. For applications, one may refer to Ragab and Green (1984) and the papers referred therein.

Recurrence relations between single moment of order statistics

In this section we establish some basic identities and a recurrence relation satisfies by the single moment and the product moment of order statistics. These results, in addition to providing some checks to estimate accuracy of the computation of moments of order statistics, can reduce the amount of direct computation of these moments of order statistics especially when the numerical approach is required.

Theorem 3.1

For $2 \le i \le n,$

$$\alpha_{i:n}^{(k)}\} = [1 + \frac{k}{(i-1)\delta}]\alpha_{i-1:n-1}^{(k)}$$

Proof

Khan *et al* (1983a) have shown that for;

For $2 \le i \le n,$

$$\alpha_{i:n}^{(k)} - \alpha_{i-1:n-1}^{(k)} = \binom{n-1}{i-1} k \int_0^{\infty} x^{k-1}[F(x)]^{i-1}[1-F(x)]^{n-i+1} dx \qquad \text{..... (3.1)}$$

Noting that

$$[F(x)]^{i-1}[1-F(x)]^{n-i+1} = \frac{x}{\delta}[F(x)]^{i-2}[1-F(x)]^{n-1} f(x)$$

In view of (2.3) we have;

$$\alpha_{i:n}^{(k)}\} = [1 + \frac{k}{(i-1)\delta}]\alpha_{i-1:n-1}^{(k)}$$

thus concluding the proof.

Theorem 3.2

for $n \ge 2,$

$$\alpha_{1:n}^{(k)} = [1 - \frac{k}{p(n-1)}]\alpha_{1:n-1}^{(k)}$$

Proof

This follows from the result:

$$\alpha_{1:n}^{(k)} = k\int_0^{\infty} x^{k-1}(1-F(x))^n dx \qquad \text{....(3.2)}$$

given by Khan et al. (1983)

Theorem 3.3. For $k < \delta$

Proof

$$E(X^k) = \alpha_{1:1}^{(k)} = e^{-k\gamma/\delta}\Gamma\left(1-\frac{k}{\delta}\right)\Gamma\left(1+\frac{k}{\delta}\right)$$

$$E(X^k) = \alpha_{1:1}^{(k_i)} = C_{1:1}\int_0^\infty u^k [F(u)]^{1-1}[1-F(u)]^{1-1} f(u)du = C_{1:1}\int_0^\infty u^k f(u)du$$

after the evaluation we have the result.

It may be mentioned here that results reported in Theorem 3.1 and 3.3 were also obtained by Ragab and Green (1984).

Theorem 3.4

for $2 \le i \le n$

$$\alpha_{i:n}^{(k-\delta)} = \frac{(n-i+1)e^{\gamma}}{(i-1)}\alpha_{i-1:n}^{(k)}$$

Proof

From (2.3),(2.4) and (3.1) we have;

$$\alpha_{i:n}^{(k)} - \alpha_{i-1:n-1}^{(k)} = \frac{k}{(i-1)\delta}\alpha_{i-1:n}^{(k)} - 1 \quad \text{....(3.3)}$$

$$= \frac{k}{(i-1)\delta e^{\gamma}}\alpha_{i:n-1}^{(k-\delta)} - 1 \quad \text{....(3.4)}$$

Equating 3.3 and 3.4 and replacing $n-1$ by n, we obtained

$$\alpha_{i:n}^{(k-\delta)} = \frac{(n-i+1)e^{\gamma}}{(i-1)}\alpha_{i-1:n}^{(k)}$$

Which is the required result.

Theorem 3.5

For $n \ge 1$

$$\alpha_{1:n}^{(k-\delta)} = \frac{n\delta e^{\gamma}}{k}\alpha_{1:n+1}^{(k-\delta)}$$

Proof

The result follows from (2.4) and (3.2).

Theorem 3.6

For $1 \le i \le n$, *and* $e^{\gamma} = 1$

$$\alpha_{i:n}^{(-k)} = \alpha_{n-i+1:n}^{(k)}.$$

Proof

Tadikamalla (1980) has shown that at $e^{\gamma} = 1$, X and $Y = 1/X$ have the same Weibull-Exponential distribution. Thus, $X_{i:n}$ and $Y_{n-i+1:n} = 1/X_{i:n}$ will have the same distribution and hence the result.

Recurrence relations between product moments of order statistics

Theorem 4.1

For $1 \le i < j \le n-1$

$$\alpha_{i,j:n}^{(l,k)} = \alpha_{i,j-1:n}^{(l,k)} + \frac{n}{(n-j+1)}\alpha_{i,j:n-1}^{(l,k)}\left(1-\frac{k}{\delta(n-j)}\right) - \frac{n}{(n-j+1)}\alpha_{i,j-1:n-1}^{(l,k)}$$

Proof
Khan *et al* (1983) have shown that for:
$1 \le i < j \le n$,

$$\alpha^{(l,k)}_{i,j:n} - \alpha^{(l,k)}_{i,j-l:n} = \frac{C_{i,j:n}}{(n-j+1)} k \int_0^\infty \int_x^\infty x^l y^{k-1} [F(x)]^{i-1} [F(y)-F(x)]^{j-i-1} [1-F(y)]^{n-j+1} f(x) dy dx.$$

writing $\{1-F(y)\}^{n-j+1}$ as ..... (4.1)

$[1-F(y)]^{n-j-1}\{[1-F(y)] - F(y)[1-F(y)]\}$

=

$[1-F(y)]^{n-j} - \frac{y}{\delta}[1-F(y)]^{n-j-1} f(y)$

In (4.1) and rearranging we get the result.

Theorem 4.2

For $1 \le i \le n-2$

$$\alpha^{(l,k)}_{i,n:n} = \alpha^{(l,k)}_{i,n-l:n} + \frac{k}{\delta} \frac{n}{(n-i-1)} \alpha^{(l,k)}_{i,n-l:n-1} - \frac{i}{(n-i+1)} (\alpha^{(l,k)}_{i+1,n:n} - \alpha^{(l,k)}_{i+1,n-l:n-1}).$$

Proof

From 4.1 $j = n$,

$$\alpha^{(l,k)}_{i,n:n} - \alpha^{(l,k)}_{i,n-l:n} = kC_{i,n:n} \int_0^\infty \int_x^\infty x^l y^{k-1} [F(x)]^{i-1} [F(y)-F(x)]^{n-i-1}$$

$*[1-F(y)] f(x) dy dx$ ....(4.2)

writing

$[F(y)-F(x)]^{n-i-1}[1-F(y)]$ as

$[F(y)-F(x)]^{n-i-2}[F(y)(1-F(y)) - F(x)(1-F(y))]$

=

$\frac{y}{\delta}[F(y)-F(x)]^{n-i-2} f(y) - F(x)[F(y)-F(x)]^{n-i-2}(1-F(y))$

in 4.2 we get result

Theorem 4.3

For Weibull-Exponential distribution with:

$1 \le i < j \le n$,

$$\alpha^{(l,k-\delta)}_{i,j:n} = \frac{\delta e^{\gamma}(n-j+2)(n-j+1)}{(n+1)k} (\alpha^{(l,k)}_{i,j:n+1} - \alpha^{(l,k-\delta)}_{i,j-l:n+1}).$$

Proof

Expressing $[1-F(y)]^{n-j+1}$ as:

$$[1-F(y)]^{n-j-1}[1-F(y)]^2 = \frac{y^{1-\delta}}{\delta e^{\gamma}}[1-F(y)]^{n-j-1} f(y)$$

in (5.1) we get

$$\alpha^{(l,k)}_{i,j:n} - \alpha^{(l,k)}_{i,j-l:n} = \frac{nk}{\delta e^{\gamma}(n-j+1)(n-j)} \alpha^{(l,k-\delta)}_{i,j:n-1},$$

And hence the result.

Note

The results obtained for the recurrence relations of single and product moment follows directly from the general results obtained by Balakrishnan *et al* (1983) and Balakrishnan *et al* (1987), interested reader may also see David, H.A. (1981).

Evaluation of means, variances and covariances

Theorem 3.1, 3.2 and 3.3 are used to calculate $\alpha_{i:n}^{(k)}$, $1 \le i \le n$. Once $\alpha_{i:n}^{(k)}$, $1 \le i \le n$ are known, theorems 3.4 and 3.5 may be used to obtained $\alpha_{i:n}^{(k-\delta)}$,. The beauty of the result is that from $\alpha_{i:n}^{(k-\delta)}$,. we can find negative and fractional moments of order statistics with some constraints, viz at k=1, δ=1.5, k-δ=-0.5 and at k=2, δ=3, k-δ= -1. Also for δ positive integer greater than one, $\alpha_{i:n}^{(k-\delta)}$,. can be used to obtain negative moments.

For calculating products moments matrix $((\alpha_{i,j:n}^{(l,k)}))$. the diagonal elements $\alpha_{i,i:n}^{(l,k)} = \alpha_{i,i+p:n}^{l+k}$,

Can be filled up first. $\alpha_{1,1:2}^{(l,k)}$,.can be obtained easily by direct numerical integration. However, at l=k, $\alpha_{1,1:2}^{(l,k)} = [\alpha_{1:1}^{(k)}]^2$. The elements $\alpha_{i,i+p:n}^{(l,k)}$, $2 \le i \le n-p-1$, $p = 1,2,\ldots\ldots,n-i-1$ are obtained from theorem 4.1. Finally, $\alpha_{1,n:n}^{(l,k)}$ are obtained from theorem 5.2. Once $\alpha_{i,i:n}^{(l,k)}$ are known $\alpha_{i,i:n}^{(l,k)} = \alpha_{i,i+p:n}^{l+k}$, can be obtained by theorem 4.3. From this, we can obtain:

$\text{Cov}(X_{i:n},X_{j:n}) = \sigma_{i,j:n} = \alpha_{i,j:n} - \alpha_{i:n}\alpha_{j:n}$,

$$\text{var}\left(\frac{X_{i:n}}{X_{j:n}}\right) = \alpha_{i,j:n}^{(2,2)} - (\alpha_{i,j:n}^{(1,1)})^2$$

these moment can also be used to find best linear unbiased estimates of location and the scale parameters of Weibull-Exponential distribution (David,1981).

Applications

In the following examples tables of the estimate μ* and σ* for a censored samples used are prepared by Balakrishnan *et al* (1987) using the numerical approach.

Illustration 1

An experiment is carried out to measure the strontium-90 concentrations in samples of milk. The test substances suppose to contain 9.22 Pico curies per liter. Ten measurements were taken, but because of relatively large measurement error known to exist at the extremes, the two smallest and the three largest observations were censored. The remaining five observations, were arranged in increasing order and are presented below.

8.2, 8.4, 9.1, 9.8, 9.9. (Shoukri *et al* 1988);
in this case we have n=10, i=2, j=3
let us assume that the above censored samples had come from a Weibull Exponential population with unknown mean μand variance σ^2 .for Weibull-Exponential distribution , the variances and covariances of μ* and σ* have been tabulated by Balakrishnan *et al* (1987). They prepared more exhaustive tables covering sample sizes $n = 2(1)25(5)40$ and all possible choices of i and j. from this tables, we obtained the BLUEs of μ and σ to be:

$\mu^* = 0.15237(8.2)+0.13241(8.4)+0.15377(9.1)+0.16153(9.8)+0.39992(9.9)= 9.3032$
and
$\sigma^* = -0.93064(8.2)-0.17440(8.4)-0.04984(9.1)+0.07908(9.8)+1.07581\,(9.9) = 1.8758$
and the standard errors of the above estimates are obtained to be:
$\text{S.E}(\mu^*) = 1.8758(0.09921)^{1/2} = 0.5908$ and $\text{S.E}(\sigma^*) = 1.8758(0.18338)^{1/2} = 0.8033$

Illustration 2

Let us consider the following data which represent the failure times in minutes, for a specific type of electrical insulation in an experiment in which the insulation was subjected to continuous increasing voltage stress.

12.3 21.8 24.4 28.6 43.2 46.9 70.7 75.3 95.5 98.1 138.6

here the largest observation was censored because the experiment stopped as soon as the eleventh failure occurred.

We assumed for the data above a two parameters Weibull-Exponential distribution.

By using the table prepared by Balakrishnan *et al* (1987). We compute the BLUEs of μ and σ to be

$\mu^* = 4.847511$ and $\sigma^* = 47.432552$.

using the above estimates, we obtain the BLUE of the expected failure times as

$\mu^* + \sigma^* \ln 4 = 4.847511 + 47.432552 \ln 4 = 70.60299$min.

and the standard error of this estimate is obtained to be

$\sigma^*[0.02440 + 0.07282(\ln 4)2 - 2(0.01547)\ln 4]1/2 = 0.348503\sigma^* = 16.53039$min.

these can be compared with

1. BLUEs of 70.01min and S.E of 17.58 min based on one-parameter half logistic distribution.
2. BLUEs of 71.5533 min and S.E. of 19.4965 min. based on two-parameter exponential distribution.

Conclusion

We get smaller S.E. in estimating the expected failure times in case of Weibull-exponential distribution for the failure time data compared to others above. The reason for the facts that a Weibull-exponential distribution fits the data better than others mentioned above.

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