A JUMP DIFFUSION PROCESS MODELLING OF NIGERIAN BONNY LIGHT CRUDE OIL PRICES

Ekong A.H.[*, a], Olayiwola O.M.[1, a], Igbalajobi, M. M.[2, a]
[a]Department of Statistics, Federal University of Agriculture Abeokuta, P.M.B 2240, Abeokuta, Ogun State, Nigeria
[1]olayiwolaom@funaab.edu.ng, [2]margaret.igbalajobi@gmail.com,
[*]corresponding author: anieekong@outlook.com

Abstract

Many attempts have been made to study and model the dynamics of the Nigeria crude oil production but not many recent works in the literature have employed stochastic differential equations in modelling crude oil prices like financial derivatives. Oil differs from financial assets in that futures prices are often below spot prices, and the degree of backwardness is highly variable over time. Oil prices are volatile and deviates from the assumptions made in many commodity pricing models which is the assumption of Gaussian dynamics for the spot price. From the foregoing, this study sought to further examine the stochastic volatility present in monthly Nigeria Bonny light crude oil prices and extend the Merton jump diffusion (MJD) model to capture the discontinuities in the trajectories of the prices as a jump process, while capturing statistical features present in the historical time series such as seasonality, mean reversion and dependencies among spot prices. Our approach involved modelling the Poisson intensity in the Levy process as exponentially decaying function. Our approach was compared with Gaussian diffusion processes of geometric Gaussian motion, Ornstein-Uhlenbeck stochastic process and its extension with mean reverting property as well as with MJD process. Simulations and application to Nigeria's Bonny light crude oil prices were used to compare the performances of the models. The MJD and our approach were very close in their parameters estimates as against the Gaussian diffusion processes and were preferred to them as evidenced by the AIC, BIC and log-likelihood. For the crude oil price modelling, the comparisons also favoured our approach over MJD process.

Keywords: jump process, crude oil prices, stochastic differential equation, nonconstant intensity, volatility

1. Introduction

In this study, the stochastic differential equations (SDEs) are applied to model the dynamics of the Nigeria crude oil production in Nigeria. Stochastic differential equations are mathematical equations that describe and model the random noise in a system. Nigeria, a country in Africa is heavily dependent on the production of oil for the sustenance of her economy, there has been successive governments trying to diversify the economy of the country from a mono-economy to poly-economy. However, in the midst of all the efforts crude oil has still remain the number one foreign earner for the country, which means that any infinitely small movement in the world oil production or prices affect the economy of Nigeria to no small measure. Being a global commodity, the price of Nigeria's Bonny light crude is affected by global happenings and news, for instance the tensions and conflicts in the recent times in Europe, Ukraine and Russia and that in the Middle East between Israel and Palestine have impacted the price of the Bonny light. In April of 2024, the price of Nigeria's Bonny Light, yesterday, rose by 2.2 per cent to $88.97 per barrel in the global oil market following increased tension in the Middle East resulting from Israel's airstrike on Iran's embassy in Damascus, Syria. This increase amounted to excess revenue of $11 per barrel for Nigeria as excess as the nation's 2024 budget was based on $77.96 per barrel and 1.78 million barrels per day (Vanguard Newspaper of 3rd April, 2024). In the same period following Ukrainian attack on Russian refineries, the Bonny light traded as $91.37 a barrel with extra revenue of $13.71 per barrel (Nairametrics, 2024). Bonny Light price was reported at $85.57 per barrel in 01 Aug 2024 and this recorded an increase from the previous number of $84.290 per barrel for 31 Jul 2024 (CEIC Data, 2024).

Many attempts have been made to study and model the dynamics of the Nigeria crude oil production using many techniques such as autoregressive integrated moving average (ARIMA) (Eriga (2013), Omekara et al. (2015), Fatoki et al. (2017), Leneenadogo and Lebari (2019), Acha et al., (2023), Suleiman et al., (2023)), seasonal ARIMA (Etaga et al., (2020)), buys-ballot modelling (Okororie et al. (2013)), machine learning models of artificial neural network (ANN) and random forest (RF) (Obite et al., (2021)), and Bartholomew et al., (2021) extended Obite et al., (2021) by adding fuzzy time series (FTS) Model. Gaspera and Mbwambob (2023) used ARIMA to analyse Tanzania Crude Oil Prices. Loera et al., (2021) assessed the relationship between the crude oil and gasoline spread and crude oil price using first-order Markov chain simulations. Gunarto et al., (2020) and Hendrawaty et al., (2021) used GARCH(1,1) models to fit daily prices of crude oil, with Hendrawaty et al., (2021) adding AR(1) to the GARCH(1,1). Adavi et al., (2021) analysed the dynamics of crude oil price volatility in Nigeria using a symmetric and asymmetric GARCH models. Monday and Abdulkadir (2020) added a parameter to ARCH model for the volatility of crude oil price in Nigeria. Usoro and Ekong (2022) fitted bivariate ARCH and GARCH models to the Nigeria crude oil price and production volatilities. AlGounmeein and Ismail (2021) applied the GARCH and the Autoregressive Fractionally Integrated Moving Average (ARFIMA) the Brent crude oil price. Lu et al. (2021) fitted a long short-term Memory Network (LSTM) for crude oil price. Obite et al., (2021) applied support vector regression (SVR), RF, ANN and Deep Neural Network (DNN) to Brent crude oil price. Rodhan and Jaaz (2021) analysed WTI crude oil price using ARIMA, Ng'ang'a and Oleche (2022) compared different GARCH models on Brent crude oil prices. Abdollahi and Ebrahimi (2020) used AFRIMA, adaptive neuro fuzzy inference system and the markov-switching models to model the Brent crude oil price. Bollapragada et al., (2021) forecasted the price of crude oil using a target capacity utilization rule recursive simulation model. Bildirici et al., (2021) used orthogonal matrix lie groups and algebras with LSTM to analyse WTI prices.

Oil differs from financial assets for example, oil futures prices are often below spot prices, and the degree of backwardness is highly variable over time. Also, the mean-reversion for oil prices can be assumed. An abnormally high price should induce higher cost for more producers to enter the market, increasing supply and ultimately causing the price to decrease, and conversely, an abnormally low price will drive many producers to leave the market, decreasing supply. Also, higher prices may make consumers to become more energy efficient or seek alternative energy sources, and vice versa when prices are low. Assuming that the risk premium is not time-varying in a way that offsets the risk-neutral mean-reversion, this will result in the property that volatility of futures prices decreases with maturity (Hughen, 2010). Oil prices are volatile and deviates from the assumptions made in many of the earlier commodity pricing models in the literature, which is the assumption of Gaussian dynamics for the spot price. Hence, the need for more stochastic volatility modelling of such assets as oil.

Hitherto, exploring the frontiers of research in this direction, not many recent works in the literature have employed stochastic differential equations in modelling crude oil prices especially in the current prevailing global climate, like there are vast literatures on financial derivatives. One recent work on SDEs for modelling financial derivatives is Gray et al., (2021), where they applied exponential power jump diffusion to model credit risk. The credit risk was assumed to have dynamics that is a combination of a diffusion process and a jump process driven by an exponential power distribution, where the diffusion component was modelled by a geometric Brownian process. Other application of SDEs to oil prices includes Nwafor and Oyedele (2017) who used the geometric Brownian motion (GBM) to model the behaviour of crude oil price, Onyeka-Ubaka and Okafor (2018) compared a jump diffusion Merton model with GARCH and AR(2) on crude oil price concluding that the jump process out-performed the other two models for estimating the drift. Gong and Wang (2022) extended the constant elasticity of variance (CEV) model to constant volatility elasticity (CVE) model and variable volatility elasticity (VVE) model to capture stochastic volatility of financial assets. Merton's model has been applied to analyse crude oil prices using the *yuima* R package (Ogbogbo, 2018, 2019). Heston stochastic volatility model has also been used to analyse West Texas Intermediate (WTI) crude oil price (Dondukova and Liu (2021), Mwanakatwe et al., (2023)). Goard and AbaOud (2023) applied a single one-factor nonlinear stochastic process to oil prices. Ajlouni and Alodat (2021) forecasted monthly gasoline prices in Jordan by applying Gaussian process regression with deterministic drift function using Bayesian approach.

From the foregoing, this study sought to further examine the stochastic volatility present in monthly Nigeria Bonny light crude oil prices for the period of January 2006 to April 2024 and extend the Merton jump diffusion (MJD) model (Merton, 1976) to capture the discontinuities in the trajectories of the prices as a jump process, particularly an inhomogeneous compound Poisson process, while capturing statistical features present in the historical time series such as seasonality, mean reversion and dependencies among spot prices. Our approach, unlike Gary et al., (2021) where it was an extension of geometric Brownian process to exponential power jump diffusion, involved modelling the Poisson intensity as exponentially decaying. We compared our proposed model to some of those which assumed Gaussian dynamics in their approaches as well as with the MJD model using model comparison measures Akaike information criterion (AIC), Bayesian information criterion (BIC) and log-likelihood as well as accuracy measures.

2. Methodology

2.1 The Lévy Processes

The GBM was based on the assumption that the assets are continuously modelled, with Brownian motion representing the noise. In real life, it has been observed in the literature that the dynamics of assets contains discontinuities, as especially for oil prices (Hughen, 2010). We aim to develop an extension to Merton (1976) model which includes the possibility of jumps with exponentially decaying intensity.

Lévy processes constitutes an important family of stochastic processes, which includes Brownian motion as the only one that is continuous.

Definition of Lévy Process: (Nunno et al., 2009)

A Lévy process L_t is a stochastic process on $(\Omega, \mathcal{F}, P)$ with the following properties:

1. $L_0 = 0$, P –a.s.
2. L_t has independent monotone increments, that is, for $t_0 \leq t_1 \leq t_2 \leq \cdots$ we have that the random variables $L_{t_0}, L_{t_1} - L_{t_0}, L_{t_2} - L_{t_1}, \cdots$ are independent.

3. L_t has stationary increments, i.e., for all $s < t$ we have that $L_t - L_s$ has the same distribution as L_{t-s}.
4. L_t is stochastically continuous, i.e., for all $\epsilon < 0$, $\lim_{h\to 0} P(|L_{t+h} - L_t| \geq \epsilon) = 0$.
5. L_t has càdlàg paths, i.e., the trajectories are right-continuous with left limits.

Unlike the Brownian motion definition, there is not property of normal increments with the Lévy process. Property 4 implies that at any time t, the probability of a jump equals zero, i.e. we can not have jumps at given times. The last property in the definition can be assumed without loss of generality because it can be shown that every Lévy processes has a càdlàg version a.s., which is also a Lévy process. Hence, we can see that Brownian motion satisfies the requirements of a Lévy process. Another example of Lévy process is the Poisson process N_t given by

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \qquad t \geq 0$$

where $\lambda > 0$ is the intensity of the process. Moreover, a compound Poisson process X_t is a process that sums a number of i.i.d. jumps sizes Y_i over a Poisson process N_t,

$$X_t = \sum_{i=1}^{N_t} Y_i \ , \qquad t \geq 0$$

where $\lambda > 0$ is the intensity and N_t is independent of Y_i. The compound Poisson process is e.g. widely used in property insurance to model the total claim amount in a portfolio, with the Y_i's representing the individual claim amounts and N_t the number of claims in the portfolio.

Lévy measure (Cont and Tankov, 2004)

Let $(L_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}$. The measure v on $\mathbb{R}$ defined by

$$v(A) = E[\#t \in [0,1]: \Delta L_t \neq 0,\ \ \Delta L_t \in A], \qquad A \in \mathfrak{B}(\mathbb{R})$$

is called the Lévy measure of L. That is, the Lévy measure denotes the expected number of jumps, per unit time, that belongs to A.

Next, we state the Itô-Lévy decomposition Theorem without proof for which the proof can be found in Cont and Tankov (2004).

Itô-Lévy decomposition (Øksendal and Sulem, 2019)

If $(L_t)_{t\geq 0}$ is a Lévy process, then it has the decomposition

$$L_t = \alpha t + \sigma B_t + \int_{|z|<R} z\widetilde{N}(t, dz) + \int_{|z|<R} zN(t, dz) \qquad (1)$$

for some constants $\alpha, \sigma \in \mathbb{R}$ and $R \in [0, \infty]$. Moreover, $\widetilde{N}(t, dz) = N(t, dz) - v(dz)dt$ is the compensated Poisson random measure of L_t and B_t is a Brownian motion which is independent of $N(t, dz)$ (Øksendal and Sulem, 2019).

The Itô-Lévy decomposition states that every Lévy process can be decomposed into a continuous Brownian motion with drift, a term incorporating the jumps that are smaller than some constant R and a term representing the jumps that are bigger or equal to R. The constant R can be chosen as small as we want, but since the case of infinitely many small jumps, i.e. $\int_{|z|<R} |z| dz = \infty$, could occur we need to compensate the Poisson random measure $N(dt, dz)$ around 0. Hence, the introduction of the compensated Poisson random measure $\widetilde{N}(t, dz)$, which can be shown to be a martingale. Since every Lévy process can be expressed by means of (1), we have that for every Lévy process there exists constants α and σ^2, together with a positive measure v, that uniquely determines its distribution. This triplet (α, σ^2, v) is often called the characteristic triplet of the Lévy process (Cont and Tankov, 2004).

The Itô formula for Itô-Lévy processes

Given the stochastic process of the form

$$X(t) = X(0) + \int_0^t \alpha(s,\omega) ds + \int_0^t \sigma(s,\omega) dB(s) + \int_0^t \int_{\mathbb{R}} \gamma(t,z,\omega) \widetilde{N}(t, dz) \tag{2}$$

where

$$\widetilde{N}(t, dz) = \begin{cases} N(dt,dz) - v(dz)dt & \text{if } |z| < R \\ N(dt,dz) & \text{if } |z| < R \end{cases}$$

for some $R \in [0, \infty]$. This Itô-Lévy process can be rewritten in short form as

$$dX(t) = \alpha(s)ds + \sigma(s)dB(s) + \int_{\mathbb{R}} \gamma(t,z) \widetilde{N}(t, dz) \tag{3}$$

Itô formula for Itô-Lévy processes

Suppose we have an Itô-Lévy process $X(t) \in \mathbb{R}$ of the form (3) where

$$\widetilde{N}(t, dz) = \begin{cases} N(dt,dz) - v(dz)dt & \text{if } |z| < R \\ N(dt,dz) & \text{if } |z| < R \end{cases}$$

for some $R \in [0, \infty]$. Further, let $f \in C^2(\mathbb{R}^2)$ and define $Y(t) = f\big(t, X(t)\big)$. Then $Y(t)$ is an Itô-Lévy process and

$$dY(t) = \frac{\partial f}{\partial t}\big(t, X(t)\big)dt + \frac{\partial f}{\partial x}\big(t, X(t)\big)[\alpha(t,\omega)dt + \sigma(t,\omega)dB(t)]$$

$$+ \frac{1}{2}\sigma^2(t,\omega)\frac{\partial^2 f}{\partial x^2}\big(t, X(t)\big)$$

$$+ \int_{|z|<R} \left\{ f\big(t, X(t^{-1}) + \gamma(t,z)\big) - f\big(t, X(t^{-1})\big) - \frac{\partial f}{\partial x}\big(, X(t^{-1})\big)\gamma(t,z) \right\} v(dz)dt$$

$$+\int_{\mathbb{R}} \{f(t, X(t^{-1}) + \gamma(t,z)) - f(t, X(t^{-1}))\}\tilde{N}(dt, dz)$$

2.2 Modelling Approach for Bonny light Crude Oil price

Let Q_t be the crude oil price at time t, the log-return of Q_t, $\ln\left(\frac{Q_t}{Q_0}\right)$ is modelled with MJD as an exponential Lévy process L_t such that (Merton, 1976)

$$\ln\left(\frac{Q_t}{Q_0}\right) = L_t = \left(\alpha - \frac{\sigma^2}{2} - \lambda\bar{k}\right)t + \sigma B_t + \sum_{i=1}^{N_t} K_i \ . \quad (4)$$

where B_t is a standard Brownian motion process, the term $\left(\alpha - \frac{\sigma^2}{2} - \lambda\bar{k}\right)t + \sigma B_t$ is a Brownian motion with drift process and $\sum_{i=1}^{L_t} K_i$ is a compound Poisson jump process, the Poisson process dN_t with intensity λ causes the price to jump randomly and the mean of the relative price jump is $\bar{k} \equiv E(k_i - 1) = e^{\gamma + \frac{\delta^2}{2}} - 1$ with variance $E([k_i - 1 - E(k_i - 1)^2]) = e^{2\gamma+\delta^2}(e^{\delta^2} - 1)$, (Matsuda, 2004).

We extend (4) to a model we shall herein refer to as eMJD, by adding compound Poisson process with exponential decaying intensity, i.e., the random jumps follow a Poisson process characterized by its intensity function $\eta(t) = \int_0^t \lambda(s)ds$ and has the distribution

$$P(N_t = r) = e^{-\eta(t)}\frac{\eta(t)^r}{r!}, \qquad r = 0,1,2, \ldots$$

where the intensity is function of time $\lambda = \lambda(t)$. Hence, we define the exponential decaying intensity as $\lambda_t = \beta \exp(-\lambda t)$.

Suppose in the small time interval the asset price jumps from Q_t to kQ_t. So the percentage change in the asset price caused by the jump is

$$\frac{dQ_t}{Q_t} = \frac{kQ_t - Q_t}{Q_t} = k - 1$$

where $\ln(k) \sim i.i.d.\, N(\gamma, \delta^2)$ and means that

$$E(k) = e^{\gamma + \frac{\delta^2}{2}} \quad \text{and} \quad E([k - E(k)^2]) = e^{2\gamma+\delta^2}(e^{\delta^2} - 1)$$

Since if $\ln x \sim N(a, b)$, then $x \sim \log\text{normal}\ (e^{a+\frac{b^2}{2}}, e^{2a+b^2}(e^{b^2} - 1))$.

So the SDE take the form

$$\frac{dQ_t}{Q_t} = (\alpha - \beta \exp(-\lambda t)\ k)dt + \sigma dB_t + (k - 1)dL_t \quad (5)$$

where α is the instantaneous expected return on the asset, σ is the instantaneous volatility of the oil price return conditional on that jump does not occur. The expected relative rate change $E\left(\frac{dQ_t}{Q_t}\right)$ from the jump part dL_t in the time interval dt is $\beta\exp(-\lambda t)\ \bar{k}dt$. This is why the instantaneous expected return on the asset αdt is adjusted by $-\beta\exp(-\lambda t)\ \bar{k}dt$ in the drift term of the jump-diffusion process to make the jump part an unpredictable innovation

$$E\left(\frac{dQ_t}{Q_t}\right) = E\left[\left(\alpha - \beta\exp(-\lambda t)\ \bar{k}\right)dt\right] + E[\sigma dB_t] + E[(k-1)dL_t]$$

$$= \left(\alpha - \beta\exp(-\lambda t)\ \bar{k}\right)dt + 0 + (b + \lambda t)\ \bar{k}dt = \alpha dt$$

From equation (5),

$$dQ_t = (\alpha - \beta\exp(-\lambda t)\ k)Q_t dt + \sigma Q_t dB_t + (k-1)Q_t dL_t \tag{6}$$

Cont and Tankov (2004) give the Itô formula for the jump-diffusion process as

$$df(X_t,t) = \frac{\partial f(X_t,t)}{\partial t}dt + b_t\frac{\partial f(X_t,t)}{\partial x}dt + \frac{\sigma_t^2}{2}\frac{\partial^2 f(X_t,t)}{\partial x^2}dt + \sigma_t\frac{\partial f(X_t,t)}{\partial x}dB_t + [f(X_{t-} + \Delta X_t) - f(X_{t-})],$$

where b_t corresponds to the drift term and σ_t corresponds to the volatility term of a jump-diffusion process

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s + \sum_{i=1}^{L_t}\Delta X_i$$

Following same we have

$$d\ln Q_t = \frac{\partial \ln Q_t}{\partial t}dt + \left(\alpha - \beta\exp(-\lambda t)\ \bar{k}\right)Q_t\frac{\partial \ln Q_t}{\partial Q_t} + \frac{\sigma^2 Q_t^2}{2}\frac{\partial^2 \ln Q_t}{\partial Q_t^2}dt + \sigma Q_t\frac{\partial \ln Q_t}{\partial Q_t}dB_t + [\ln kQ_t - \ln Q_t] \tag{7}$$

$$d\ln Q_t = \left(\alpha + \lambda t(\ln\beta)\ \bar{k}\right)Q_t\frac{1}{Q_t}dt + \frac{\sigma^2 Q_t^2}{2}\left(-\frac{1}{Q_t^2}\right)dt + \sigma Q_t\frac{1}{Q_t}dB_t + [\ln k + \ln Q_t - \ln Q_t]$$

$$= \left(\alpha + \lambda t(\ln\beta)\ \bar{k}\right)dt - \frac{\sigma^2}{2}dt + \sigma dB_t + \ln k$$

$$\ln Q_t - \ln Q_0 = \left(\alpha - \frac{\sigma^2}{2} + \lambda t(\ln\beta)\ \bar{k}\right)(t-0) + \sigma_t(B_t - B_0) + \sum_{i=1}^{L_t}\ln k_i$$

$$\ln Q_t = \ln Q_0 + \left(\alpha - \frac{\sigma^2}{2} + \lambda t(\ln\beta)\ \bar{k}\right)(t-0) + \sigma_t(B_t - B_0) + \sum_{i=1}^{L_t}\ln k_i$$

$$\ln Q_t = \ln Q_0 + \left(\alpha - \frac{\sigma^2}{2} + \lambda t(\ln \beta)\,\bar{k}\right)(t-0) + \sigma_t B_t + \sum_{i=1}^{L_t} \ln k_i$$

$$\exp(\ln Q_t) = \exp\left\{\ln Q_0 + \left(\alpha - \frac{\sigma^2}{2} + \lambda t(\ln \beta)\,\bar{k}\right)(t-0) + \sigma_t B_t + \sum_{i=1}^{L_t} \ln k_i\right\}$$

$$Q_t = Q_0 \exp\left\{\beta\left(\alpha - \frac{\sigma^2}{2}\lambda t\,\bar{k}\right)t + \sigma_t B_t\right\}\exp\left(\sum_{i=1}^{L_t} \ln k_i\right)$$

$$Q_t = Q_0 \exp\left\{\beta\left(\alpha - \frac{\sigma^2}{2}\lambda t\,\bar{k}\right)t + \sigma B_t\right\}\prod_{i=1}^{L_t} k_i$$

$$Q_t = Q_0 \exp\left\{\beta\left(\alpha - \frac{\sigma^2}{2}\lambda t\,\bar{k}\right)t + \sigma B_t + \sum_{i=1}^{L_t} \ln k_i\right\} \tag{8}$$

Using the previous definition of the log rate jump size $\ln k_i = K_i$

$$Q_t = Q_0 \exp\left\{\beta\left(\alpha - \frac{\sigma^2}{2}\lambda t\,\bar{k}\right)t + \sigma B_t + \sum_{i=1}^{L_t} K_i\right\} \tag{9}$$

This implies that Q_t is an exponential Lévy model $Q_t = Q_0 e^{L_t}$ with a compound Poisson jump part given as

$$L_t = \beta\left(\alpha - \frac{\sigma^2}{2}\lambda t\,\bar{k}\right)t + \sigma B_t + \sum_{i=1}^{L_t} K_i$$

We note that the compound Poisson jump process $\prod_{i=1}^{L_t} k_i = 1$ if $L_t = 0$ or positive and negative jumps cancel each other out.

In the Black-Scholes case, log return $\ln(Q_t/Q_0)$ is normally distributed (Black and Scholes, 1973)

$$Q_t = Q_0 \exp\left\{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma B_t\right\}$$

$$\ln(Q_t/Q_0) \sim N\left(\left(\alpha - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right)$$

Merton (1976) posited that the existence of compound Poisson jump process makes log return non-normal, which enables the probability density of log return $x_t = \ln(Q_t/Q_0)$ to be obtained as a quickly converging series of the following form

$$P(x_t \in A) = \sum_{i=0}^{\infty} P(L_t = i) P(x_t \in A | L_t = i)$$

$$P(x_t) = \sum_{i=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^i}{i!} N\left(x_t; \beta\left(\alpha - \frac{\sigma^2}{2}\lambda t\, \bar{k}\right)t + i\gamma, \sigma^2 t + i\delta^2\right) \quad (10)$$

where

$$N\left(x_t; \beta\left(\alpha - \frac{\sigma^2}{2}\lambda t\, \bar{k}\right)t + i\gamma, \sigma^2 t + i\delta^2\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 t + i\delta^2}} \exp\left[-\frac{x_t - \left\{\beta\left(\alpha - \frac{\sigma^2}{2}\lambda t\, \bar{k}\right)t + i\gamma\right\}}{2(\sigma^2 t + i\delta^2)}\right]$$

The term $P(L_t = i) = \frac{e^{-\lambda t}(\lambda t)^i}{i!}$ is the probability that the asset price jumps i times during the time interval of length t and

$$P(x_t \in A | L_t = i) = N\left(x_t; \beta\left(\alpha - \frac{\sigma^2}{2}\lambda t\, \bar{k}\right)t + i\gamma, \sigma^2 t + i\delta^2\right)$$

is the Black-Scholes normal density of log-return assuming that the asset price jumps i times in the time interval of t. Therefore, the log-return density as in the MJD model can be interpreted as the weighted average of the Black-Scholes normal density by the probability that the asset price jumps i times.

The characteristic function of the model can be calculated by Fourier transform of the log-return density function with parameters $(a, b) = (1,1)$

$$\phi(\omega) = \int_{-\infty}^{\infty} \exp(i\omega x_t)\, P(x_t) dx_t \quad (11)$$

$$= \exp\left[\beta \exp(-\lambda t) \exp\left\{\frac{1}{2}\omega(2i\gamma - \delta^2\omega)\right\} - (b + \lambda t)\left(1 + i\omega\bar{k}\right) - \frac{1}{2}t\omega\{-2i\alpha + \sigma^2(i + \omega)\}\right]$$

$$= \exp\left[\beta \exp(-\lambda t) \exp\{\omega i\gamma - \delta^2\omega^2\} - \left(b + bi\omega\bar{k} + \lambda t + i\omega\bar{k}\lambda t\right) + i\alpha t\omega - \frac{1}{2}t\omega\sigma^2 i - \frac{1}{2}ti\omega^2\right]$$

$$= \exp\left[\beta \exp(-\lambda t) \exp\{\omega i\gamma - \delta^2\omega^2\} - b - bi\omega\bar{k} - \lambda t - i\omega\bar{k}\lambda t + i\alpha t\omega - \frac{1}{2}t\omega\sigma^2 i - \frac{1}{2}ti\omega^2\right]$$

$$= \exp\left[\beta \exp(-\lambda t) \exp\{\omega i\gamma - \delta^2\omega^2\} - \frac{i\omega}{2}\left(2b\bar{k} + 2\bar{k}\lambda t - 2\alpha t - t\sigma^2\right) - \frac{1}{2}ti\omega^2 - (b + \lambda t)\right]$$

$$= \exp\left[\beta\exp(-\lambda t)\exp\{\omega i\gamma - \delta^2\omega^2\} - (b+\lambda t) - \frac{i\omega}{2}\left(2b\bar{k} + 2\bar{k}\lambda t - 2\alpha t - t\sigma^2\right) - \frac{1}{2}ti\omega^2\right]$$

$$= \exp\left[\beta\exp(-\lambda t)\exp(\{\omega i\gamma - \delta^2\omega^2\} - 1) - \frac{i\omega}{2}\left(2b\bar{k} + 2\bar{k}\lambda t - 2\alpha t - t\sigma^2\right) - \frac{1}{2}ti\omega^2\right]$$

$$= \exp\left[\beta\exp(-\lambda t)\exp(\{\omega i\gamma - \delta^2\omega^2\} - 1) - i\omega\left(b\bar{k} + \bar{k}\lambda t - \alpha t - \frac{t\sigma^2}{2}\right) - \frac{1}{2}ti\omega^2\right]$$

$$= \exp\left[\beta\exp(-\lambda t)\exp(\{\omega i\gamma - \delta^2\omega^2\} - 1) - i\omega\left(-\alpha t - \frac{t\sigma^2}{2} + \beta\exp(-\lambda t)\,\bar{k}\right) - \frac{1}{2}ti\omega^2\right]$$

$$= \exp\left[\beta\exp(-\lambda t)\exp(\{\omega i\gamma - \delta^2\omega^2\} - 1) + i\omega\left(\alpha t - \frac{t\sigma^2}{2}\beta\exp(-\lambda t)\ \bar{k}\right) - \frac{1}{2}ti\omega^2\right]$$

Let

$$\psi(\omega) = \beta\exp(-\lambda t)\exp(\{\omega i\gamma - \delta^2\omega^2\} - 1) + i\omega\left(\alpha t - \frac{t\sigma^2}{2}\beta\exp(-\lambda t)\ \bar{k}\right) - \frac{1}{2}ti\omega^2 \quad (12)$$

be the characteristic exponent or cumulant generating function, where $\bar{k} \equiv e^{\gamma+\frac{\delta^2}{2}} - 1$. We then have

$$\phi(\omega) = \exp[t\psi(\omega)]$$

The characteristic exponent (12) can be alternatively obtained by substituting the Lévy measure of the model

$$\ell(dx) = \frac{\lambda}{\sqrt{2\pi\delta^2}}\exp\left\{\frac{(dx-\gamma)}{2\delta^2}\right\} = \beta\exp(-\lambda t)\,f(dx)$$

into the Lévy-Khinchin representation of the finite variation type (Matsuda (2004))

$$\psi(\omega) = \omega iv - \frac{\delta^2\omega^2}{2} + \int_{-\infty}^{\infty}\{\exp(i\omega x) - 1\}\ell(dx)$$

$$\psi(\omega) = \omega iv - \frac{\delta^2\omega^2}{2} + \int_{-\infty}^{\infty}\{\exp(i\omega x) - 1\}\beta\exp(-\lambda t)\,f(dx)$$

$$\psi(\omega) = \omega iv - \frac{\delta^2\omega^2}{2} + \beta\exp(-\lambda t)\int_{-\infty}^{\infty}\{\exp(i\omega x) - 1\}f(dx)$$

$$\psi(\omega) = \omega iv - \frac{\delta^2\omega^2}{2} + \beta\exp(-\lambda t)\left\{\int_{-\infty}^{\infty}\exp(i\omega x)f(dx) - \int_{-\infty}^{\infty}f(dx)\right\} \quad (13)$$

Since $\int_{-\infty}^{\infty}\exp(i\omega x)f(dx)$ is the characteristic function of $f(dx)$,

$$\int_{-\infty}^{\infty} \exp(i\omega x) f(dx) = \exp\left(\omega i v - \frac{\delta^2\omega^2}{2}\right)$$

Therefore,

$$\psi(\omega) = \omega i v - \frac{\delta^2\omega^2}{2} + \beta \exp(-\lambda t)\left\{\exp\left(\omega i v - \frac{\delta^2\omega^2}{2}\right) - 1\right\} \qquad (14)$$

where $v = \alpha t - \frac{t\sigma^2}{2}\beta \exp(-\lambda t)\ \bar{k}$. This corresponds to (12). According to Matsuda (2004), the Characteristic exponent (12) generates cumulants as follows

$$cum_1 = \alpha - \frac{\sigma^2}{2}\beta \exp(-\lambda t)\,\bar{k} + \frac{\lambda}{\beta}\mu$$

$$cum_2 = \sigma^2 + \delta^2\lambda + \mu^2\beta\lambda$$

$$cum_3 = \lambda(3\delta^2\mu + \mu^3)$$

$$cum_4 = \lambda(3\delta^4\mu + 6\mu^2\delta^2 + \mu^4)$$

Annualized (per unit of time) mean, variance, skewness, and excess kurtosis of the log-return density $P(x_t)$ are computed from above cumulants as follows

$$E(x_t) = cum_1 = \alpha - \frac{\sigma^2}{2}\beta \exp(-\lambda t)\left(e^{\gamma + \frac{\delta^2}{2}} - 1\right) + \frac{\lambda}{\beta}\mu$$

$$\mathrm{Var}(x_t) = cum_2 = \sigma^2 + \delta^2\lambda + \mu^2\beta\lambda$$

$$\mathrm{Skewness}(x_t) = \frac{cum_3}{(cum_2)^{3/2}} = \frac{\lambda(3\delta^2\mu + \mu^3)}{(\sigma^2 + \delta^2\lambda + \mu^2\beta\lambda)^{3/2}}$$

$$\mathrm{Kurtosis}(x_t) = \frac{cum_4}{(cum_2)^2} = \frac{\lambda(3\delta^4\mu + 6\mu^2\delta^2 + \mu^4)}{(\sigma^2 + \delta^2\lambda + \mu^2\beta\lambda)^2}$$

2.3 Quasi-maximum likelihood estimation of log-returns for the proposed exchange rate model

It has been noted that the exact maximum likelihood estimation is mostly infeasible for the statistical model (3) and (6), since the transition probability associated with *X* in (3) is not available in a closed form, hence the sought for alternative M-estimation and of interest is the Quai-maximum likelihood estimation, which is known to have the advantage of computational simplicity and robustness for model misspecification, in compensation for some amount of information loss (Masuda, 2011). We refer to Shimizu and Yoshida (2006) and Ogihara and Yoshida (2011) for details in parametric estimation of jump-diffusion processes.

For the quasi-maximum likelihood estimation applied in our study, we follow the specification in Brouste et al., (2014) by considering the diffusion process

$$dQ_t = a(Q_t, \theta_2)dt + b(Q_t, \theta_1)dW_t \qquad Q_0 = q_0,$$

with W_t being a standard Wiener process independent of the initial variable Q_0 and $(\theta_1, \theta_2) \in\subset \mathbb{R}$. Given sampled data $\boldsymbol{Q}_n = \left(Q_{t_i}\right)_{i=0,\dots,n}$ with $t_i = i\Delta_n$, $\Delta_n \to 0$ as $n \to \infty$, quasi-maximum likelihood estimator makes use of the following approximation of the true log-likelihood for multidimensional diffusions

$$\ell_n(\boldsymbol{Q}_n, \theta) = -\frac{1}{2}\sum_{i=1}^{n}\left\{\log\det\left(\Sigma_{i-1}(\theta_1)\right) + \frac{1}{\Delta_n}\Sigma_{i-1}^{-1}(\theta_1)\left[\left(\Delta Q_i - \Delta_n a_{i-1}(\theta_2)\right)^{\otimes 2}\right]\right\},$$

where $\theta = (\theta_1, \theta_2)$, $\Delta Q_i = Q_{t_i} - Q_{t_{i-1}}$, $\Sigma_i(\theta_1) = \Sigma\left(\theta_1, Q_{t_i}\right)$, $a_i(\theta_2) = a\left(Q_{t_i}, \theta_2\right)$, $\Sigma = b^{\otimes 2}$, $A^{\otimes 2} = AA'$ and A^{-1} is the inverse of A, $A[B] = \mathrm{tr}(AB)$. Then the QMLE of θ is an estimator that satisfies

$$\hat{\theta} = \arg\max_{\theta} \ell_n(\boldsymbol{Q}_n, \theta)$$

exactly or approximately.

For the requirement of consistency of the estimator $\hat{\theta}_1$, it is assumed that $\Delta_n \to 0$ as $n \to \infty$. Indeed, under this condition, $\hat{\theta}_1$ has asymptotically (mixed) normality (Uchida and Yoshida (2012), Brouste et al., (2014)). For the consistency of $\hat{\theta}_2$, when $T = n\Delta_n \to \infty$, usually ergodicity is assumed to ensure a law of large numbers and as a result the consistency of $\hat{\theta}_2$ is obtained and asymptotic normality is also established. This is so because the Fisher information for $\hat{\theta}_2$ is finite for a finite *T* and consistent estimation of θ_2 is theoretically impossible.

3. Results and Discussion

3.1 Examination of Bonny Light Crude Oil Price Time series properties

Figure 1 shows the time series plot of the Bonny light crude oil price data from the period of January 2006 to April 2024. Examining the time series properties of the data shows that the price is highly volatile. Moreover, due to the presence of skewness and fat tails in the empirical distribution of oil price returns, the series is more effectively modelled using Levy models which assign higher probabilities to sudden unexpected events unlike the Gaussian models like Black Scholes model. Figure 2 further shows the properties of the dataset as it is decomposed to the seasonal, trend and random components.

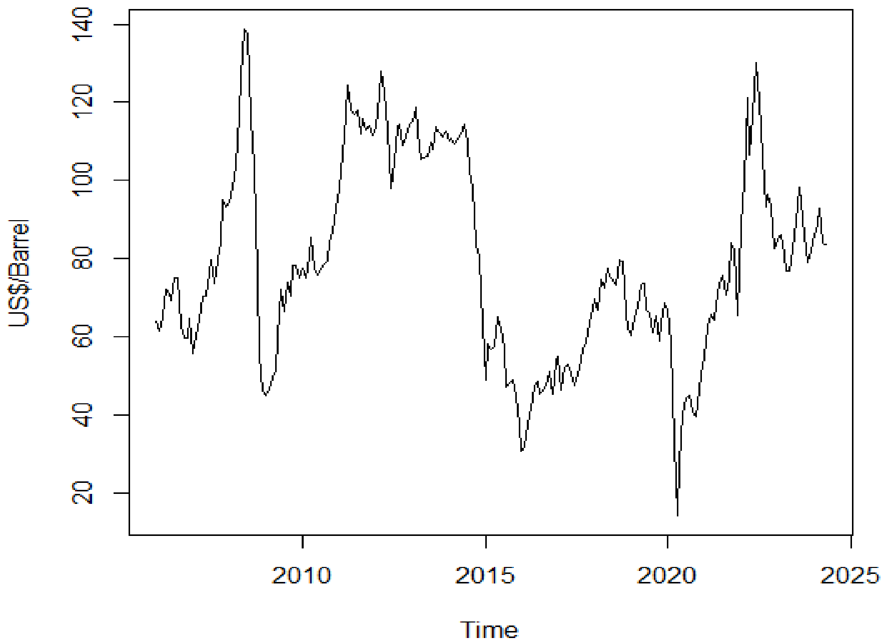

Figure 1. Time series plot of the Bonny light crude oil price data

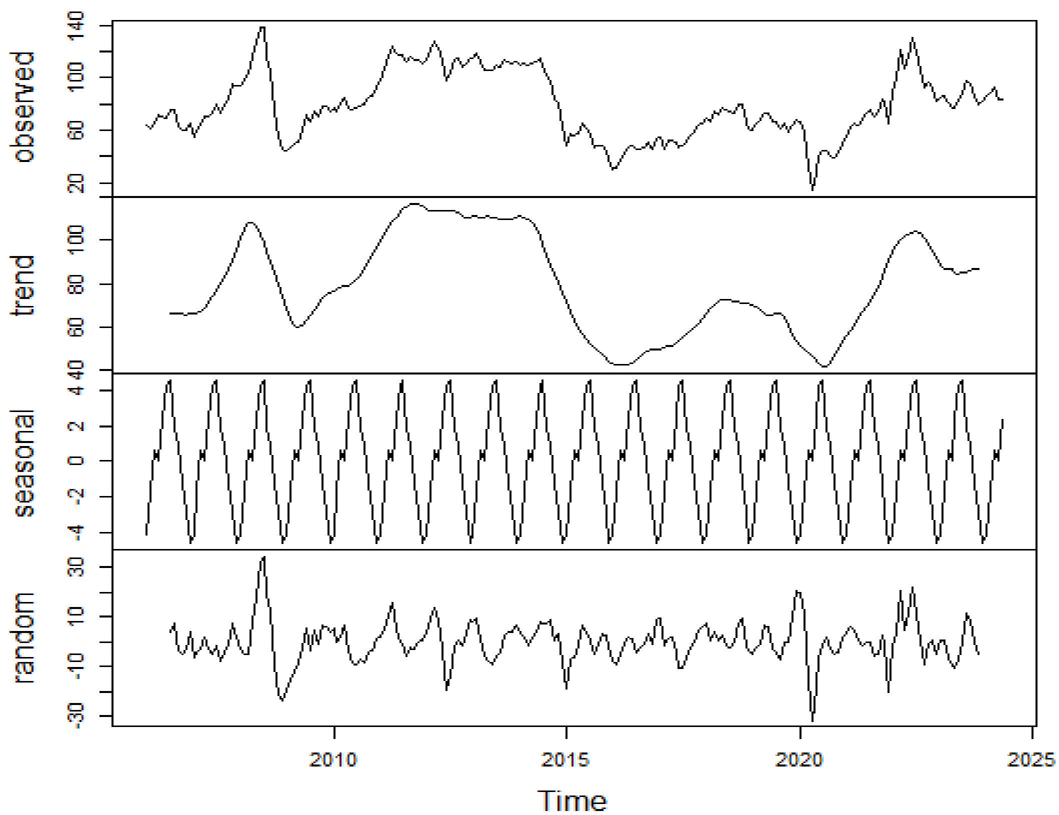

Figure 2. Additive Decomposition of the Bonny light crude oil price data

The series shows the randomness as well as seasonality in the data with no particular trend over the cumulative period, even though there have been highs and lows, period when the prices have been highest and lowest respectively. The highs and lows seem to indicate the possible jumps in the crude oil prices that seem to influence the volatility of the price in time. We investigate further these possible jumps by examining the histogram and density plots of the data as given in Figure 3. To confirm the seasonality factor in the time series, we run a dickey fuller test to check for stationarity. The augmented Dickey-Fuller test gave a Dickey-Fuller statistic of -2.4063, Lag order of 6 and a p-value of 0.4054, indicating the non-stationarity of the dataset.

As evidenced from the Figure 3 below, the Bonny light crude oil price is quite volatile as characterized by a number of jumps. Examining stationarity of the data, we shall transform the prices to log returns to better analyse and examine the parameter estimation.

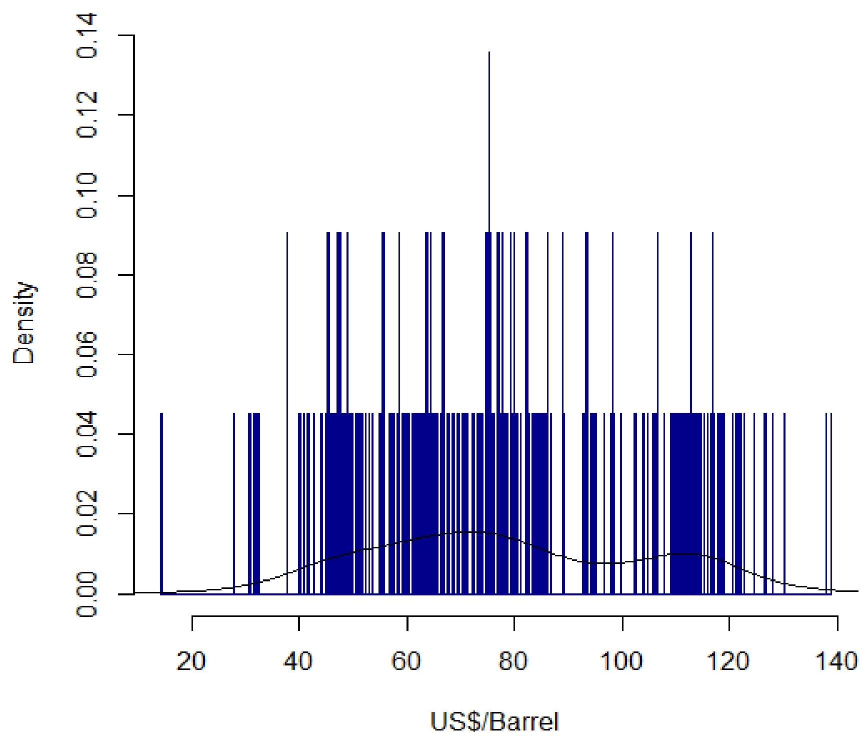

Figure 3. Histogram and Density plot of the Bonny light crude oil price data

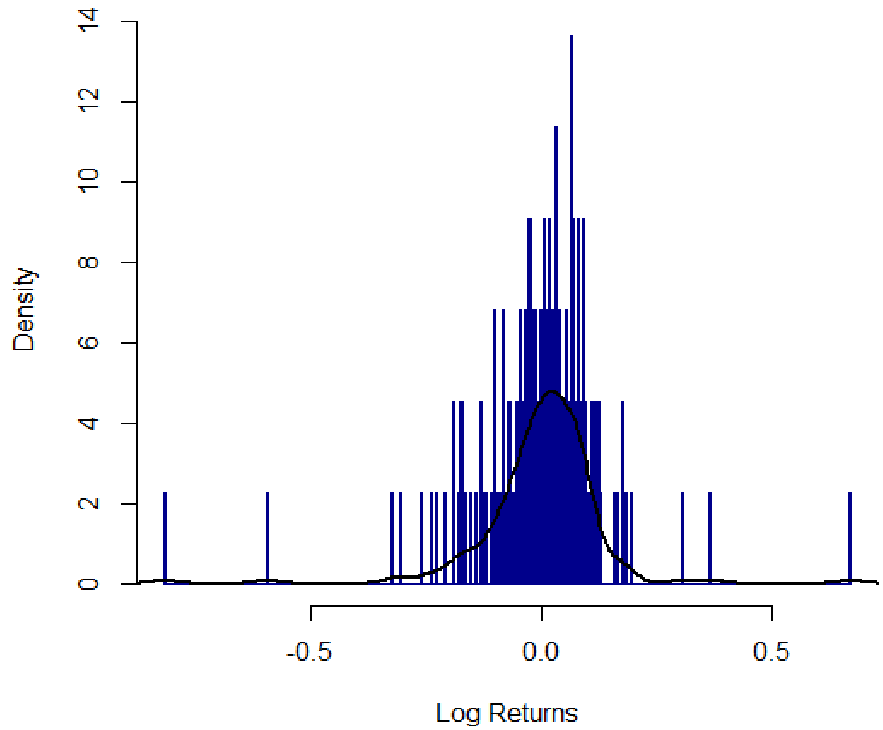

Figure 4 Histogram and Density plot of the log-returns of Bonny light crude oil price

From Figure 4 above, we see from the log-returns that there has been heightened variance in the crude oil price, with the earlier periods being characterized by relatively high prices and low levels for the most recent periods. Figure 3 shows that also shows that the crude oil price series is not Gaussian in distribution, whereas the log-return plot in Figure 4 shows an approximate Gaussian distribution with negative skewness. These properties mean that the crude oil price cannot be efficiently modelled using existing approaches the geometric Brownian motion, or the Black and Scholes model. This negative skewness and kurtosis in the log-return density of the crude oil price can be better captured in Merton jump diffusion (MJD) model with the introduction three extra parameters to the original Black and Scholes model, as noted by Matsuda (2004).

We therefore apply our extension of the Merton model to cater for the negative skewness and kurtosis, and the volatility is also captured by the inclusion of the exponential jump process with nonconstant intensity. First, since it is known from the theory that the estimation of the drift in a diffusion process strongly depends on the length of the observation interval [0, *T*], we carry out a simulation exercise to see the effects of sample size and the length of the observation interval [0, *T*] on the estimation of the eMJD model against other models under consideration.

3.2 Simulation study for Comparison of models

We simulated a stochastic differential equation diffusion process with a compound Poisson jump and exponential decaying intensity with parameter values drift = 0.1, diffusion coefficient = 0.5, beta = 10, lambda = 0.2, gamma = 2 and jump coefficient = 15. We considered sample size of simulations, n as 500, 3500 and 10000 and we took $T = n^{1/3}$, with n = 500, we have approximatively 7.94, with n = 3500, we have approximately 15.18 and with n = 10000 we have approximately 21.54. We also varied T using the values 10 and 1 each for n = 10000. The results from the simulation are given in Table 1 and Table 2. We compared eMJD approach with Gaussian diffusion processes of geometric Gaussian motion (GBM), Ornstein-Uhlenbeck (O-U) stochastic process and its extension with mean reverting property also referred to Vasicek (VAS) model and MJD process.

Table 1: Simulation Results for n = 500 and 3500

	GBM	O-U	VAS	MJD	eMJD
			n = 500; T = 7.94		
AIC	10216.66	4155.36	5708.61	3120.55	2306.60
BIC	10225.09	4163.79	5721.25	3141.62	2327.66
log like	10212.66	4151.36	5702.61	3110.55	2296.60
rmse	844.7616	172.1288	1881.2910	12.4500	12.5768
smape	1.9990	1.6196	1.9989	1.5373	1.4720
			n = 3500; T = 15.18		
AIC	13122.82	11983.31	19450.70	11580.88	11199.59
BIC	13133.44	11993.93	19466.64	11607.44	11226.16
log like	13118.82	11979.31	19444.70	11570.88	11189.59
rmse	1.4363	146.9239	4129.6580	12.4317	21.8906
smape	1.4473	1.2777	1.9995	1.3244	1.5702

We decided to use the model comparison diagnostics of Akaike information criterion (AIC), Bayesian information criterion (BIC) and log-likelihood to compare the models, while root mean squared error (RMSE) and symmetric mean absolute percentage error (SMAPE) were used to compare the parameter estimates to the specified true values. We however, point out that since the models compared do not all have the same parameterisation, our comparison of the estimated coefficients with the true values may not be very accurate, but we nonetheless anchor our comparison here on the fact that the models all estimates the drift and diffusion coefficients.

From Table 1 we observe that with sample size 500, the eMJD model had the lowest AIC, BIC, log-likelihood and SMAPE followed by MJD which had the lowest RMSE. With sample size 3500, Ornstein-Uhlenbeck model had lowest SMAPE, geometric Brownian motion had the lowest RMSE, whereas eMJD had the lowest AIC, BIC and log-likelihood and again closely followed by MJD. We also noted that the AIC, BIC and log-likelihood increased as sample size increased from 500 to 3500 for all models.

Table 2: Simulation Results for n = 10000 and Varied *T*

	GBM	O-U	VAS	MJD	eMJD
			n = 10000; T = 21.54		
AIC	93866.54	98240.60	98242.60	85364.89	69509.11
BIC	93880.96	98255.02	98264.23	85400.95	69545.16
log like	93862.54	98236.60	98236.60	85354.89	69499.11
rmse	0.2442	368.7551	521.5025	12.4351	16.1974
smape	0.7121	1.1924	1.9962	1.3222	1.5496
			n = 10000; T = 10		
AIC	76031.83	68989.43	105793.20	73301.14	46321.99
BIC	76046.25	69003.85	105814.90	73337.19	46358.04
log like	76027.83	68985.43	105787.20	73291.14	46311.99
rmse	0.8463	85.1447	1248.9450	7.1107	12.4790
smape	1.4100	1.5640	1.9984	0.8476	1.4687
			n = 10000; T = 1		
AIC	22107.41	6883.78	112921.40	22019.06	8860.02
BIC	22121.83	6898.20	112943.00	22055.11	8896.07
log like	22103.41	6879.78	112915.40	22009.06	8850.02
rmse	0.0448	3.4790	1792.3130	7.9941	12.0006
smape	0.2363	1.8310	1.9989	0.8908	1.4031

From Table 2 it is seen that again the AIC, BIC and log-likelihood increased as sample size increased from 3500 to 10000 for all models with *T* = 21.54. However, as T dropped to 10 and 1, the AIC, BIC and log-likelihood also dropped. As *T* varied from 21.54 to 10 to 1, eMJD was the preferred model to others as evidenced in the AIC, BIC and log-likelihood values reported. Hence, from the simulation results, the sample size and the length of the observation interval *T*, does affect the quasi-maximum likelihood estimation of the diffusion models with jumps, but in whichever case, the eMJD model was preferred to the models compared with in this study.

3.3 Application of models to Bonny Light crude oil price data

The models compared in the simulation study are fitted to the real dataset of Bonny light crude oil price. Table 3 presents the drift coefficient and diffusion coefficients along with AIC, BIC and log-likelihood values for each of the five models considered. We see that the MJD and eMJD models have the same values for the drift and diffusion coefficients. Comparing the models with AIC, BIC and log-likelihood from Table 3 shows eMJD with lowest of these values, giving evidence to the eMJD model as being the best fitting model for the Bonny light crude oil price.

Table 3: Model fits on Bonny Light Crude Oil Prices

	GBM	O-U	VAS	MJD	eMJD
drift	2.204272	0.748217	0.000195	4.144635	4.144635
diffusion	-1.96553	242.9948	4792.073	1.687838	1.687838
AIC	1604.690	1659.503	2922.865	1537.729	1499.390
BIC	1611.477	1666.291	2933.046	1554.697	1516.358
log like	1600.690	1655.503	2916.865	1527.729	1489.390

A look at the eMJD model fit summary statistics for the Bonny light crude oil price as shown in Table 4 shows a drift coefficient (mu) value of 4.145 and a diffusion coefficient (sigma) value of 1.688 with standard error of 0.0825. The value of the Poisson intensity parameter (lambda) was 0.001 and the rate (gamma) of the exponential intensity was 0.1. The average jump time was 0.491 and the estimated jumps was 9 with an average jump size and jump threshold of -11.487 and 15.875 respectively. We not from the structure of above model that there are periods of normal price distribution and random intermittent unusual price movements resulting in jumps. We noted from the result that for the Bonny light crude oil price studied, the diffusion was positive and significant, whereas the jump component was also positive, it was not really significant at 0.001. This means that for this dataset, the jumps were not quite as influential in the crude oil price dynamics. The high drift coefficient indicated that the presence of intermittent upward trends in the crude oil price with variable probability of the jumps. The forecast values from the eMJD model fit as shown in Figure 5, highlights the forecast jump trajectory to be consistent with the crude oil price. The implementation of the modelling and data analyses were carried out using the *yuima* package (Brouste et al., (2014) for R. The distinction between the true price trajectories and the forecast jumps in Figure 5 is shown by the jump diffusion model forecasts (circles) on the price series path.

Table 4: eMJD model fits Summary on Bonny Light Crude Oil Prices

	Coeff.	Estimate	Std. Error		
	sigma	1.68784	0.08246		
	mu	4.14464	1.84455		
	beta	2.29191	3.24089		
	lambda	0.00100	2.80578		
	gamma	0.10000	0.07070		
	Number of estimated jumps: 9				
	Average inter-arrival times: 0.080853				
	Average jump size: -11.4867				
	Standard Dev. Of jump size: 19.65021				
	Jump Threshold: 15.87451				
Summary statistics for jump times:					
Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.119	0.131	0.6706	0.4912	0.754	0.7659
Summary statistics for jump size:					
Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
-28.51	-21.9	-18.01	-11.49	-16.75	23.3

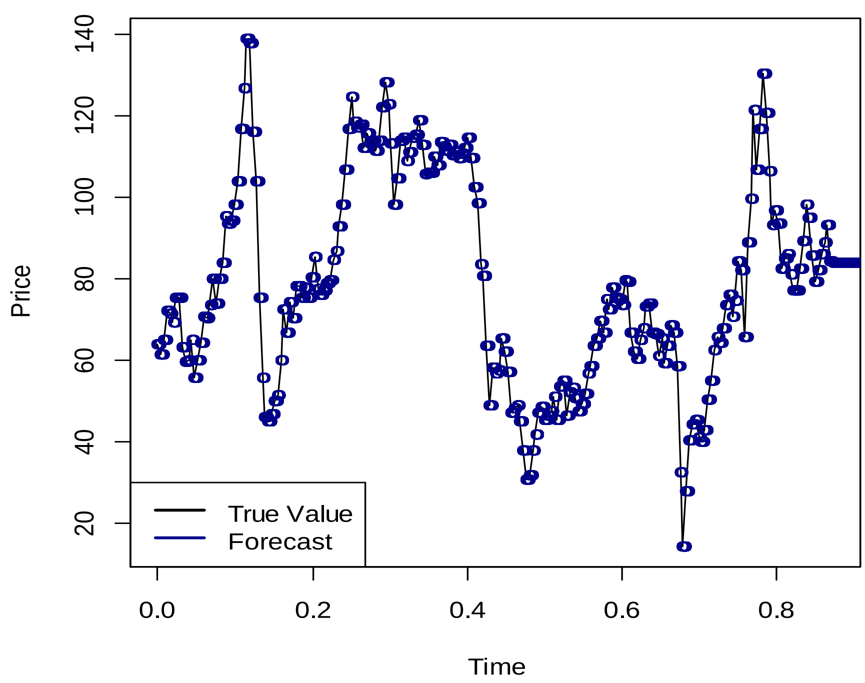

Figure 5: Plot of Original Bonny light Crude oil prices and eMJD model forecast values

4. Conclusion

We studied the price fluctuations of Nigeria's Bonny light crude oil, looking at the time series properties and the possibility of fitting an extension of the Merton jump diffusion (MJD) model to accommodate the skewness, kurtosis and non-stationarity in the crude oil price. The stochastic differential equation (SDE) with jump diffusion introduces modelling approach for the stochastic volatility in the time series. The MJD involved a compound Poisson process with constant intensity and this study examined the case of non-constant intensity using an exponential distribution for the intensity function to account for more randomness in the jump process, and referred to this extension as eMJD. We compared eMJD approach with Gaussian diffusion processes of geometric Gaussian motion (GBM), Ornstein-Uhlenbeck (O-U) stochastic process and its extension with mean reverting property also referred to Vasicek (VAS) model and MJD process. We saw that the MJD and eMJD were very close in their parameters estimates as against the Gaussian diffusion processes and were preferred to the Gaussian processes. Comparing MJD and eMJD models using AIC and BIC showed that eMJD was preferred to MJD in both the simulation study and the crude oil price. SDE with Levy jump process as noted in the literature captures more than the stochastic volatility in prices, but also captures other properties like skewness, kurtosis and jumps as seen in crude oil price studied here. Several extensions for future studies are possible to capture more time series properties and trajectories by looking at other functions for the Poisson intensity. Also, other distributions for the jump process like gamma distribution could be fitted. With the help of R packages such as *yuima* application of these approaches can be easily done by practitioners in industry to better understand the price dynamics.

Reference

Abdollahi, H. and Ebrahimi, S. B. (2020). A new hybrid model for forecasting Brent crude oil price. Energy, 200, 117520. https://doi.org/10.1016/j.energy.2020.117520.

Acha, C. K., Amalahu, C. C. and Eziokwu, C. E. (2023). Modeling and Forecasting of Nigeria Crude Oil Production. JMSS 4(1): 58-67. DOI: 10.32996/jmss.

Adavi, O. Y., Ahmad, A. H. and John, D.I. (2021). Modelling and Forecasting Volatility of Crude Oil Returns in Nigeria based on Six Error Innovations. Journal of Statistical Modeling and Analytics, Vol 3(1), 78-93. 2021

Ajlouni, S. A. and Alodat, M. T. (2021). Gaussian Process Regression for Forecasting Gasoline Prices in Jordan. International Journal of Energy Economics and Policy, 2021, 11(3), 502-509.

AlGounmeein, R. S. and Ismail, M. T. (2021). Modelling and forecasting monthly Brent crude oil prices: A long memory and volatility approach. Statistics in Transition New Series, 22(1), 29-54.Available at: https://doi.org/10.21307/stattrans-2021-002.

Bartholomew, D.C., Orumie, U.C., Obite, C.P., Duru, B.I. and Akanno, F.C. (2021). Modeling the Nigerian Bonny Light Crude Oil Price: The Power of Fuzzy Time Series. Open Journal of Modelling and Simulation, 9, 370-390. https://doi.org/10.4236/ojmsi.2021.94024

Bildirici, M., Bayazit, N.G. and Ucan, Y. (2021). Modelling Oil Price with Lie Algebras and Long Short-Term Memory Networks. Mathematics 2021, 9, 1708. https://doi.org/10.3390/math9141708

Bollapragada, R., Mankude, A. and Udayabhanu, V. (2021). Forecasting the price of crude oil. Decision (June 2021) 48(2):207–231. https://doi.org/10.1007/s40622-021-00279-5

Brouste, A., Fukasawa, M., Hino, H., Iacus, S.M., Kamatani, K., Koike, Y., Masuda, H., Nomura, R., Ogihara, T., Shimuzu, Y., Uchida M. and Yoshida, N. (2014). The YUIMA Project: A Computational Framework for Simulation and Inference of Stochastic Differential Equations. Journal of Statistical Software, Volume 57, Issue 4. Doi:10.18637/jss.v057.i04.

CEIC Data, (2024). Nigeria Crude Oil Price: Bonny Light: per Barrel. Retrieved on 05 August 2024 from https://www.ceicdata.com/en/nigeria/crude-oil-price/crude-oil-price-bonny-light-per-barrel

Cont, R. and Tankov, P. (2004). Financial Modelling with Jump Processes. Chapman & Hall/CRC Financial Mathematics Series.

Dondukova O. and Liu Y. (2021). Forecasting the Crude Oil Prices Volatility with Stochastic Volatility Models. SAGE Open July-September 2021: 1–8. https://doi.org/10.1177/21582440211026269

Eriga, F.A., 2013. Box and Jenkins ARIMA Modelling of Crude Oil Production in Nigeria. LAP Lambert Academic Publishing, Sunnyvale, USA.

Etaga, H. O., Igwebuike, E. K. and Etaga, C. N. (2020). Time Series Models of Crude Oil Production and Export in Nigeria (1999-2015). African Journal of Mathematics and Statistics Studies, Volume 3, Issue 1, 2020 (pp. 1-24).

Fatoki, O., Ilo, H.O., Ugwoke, P.O., 2017. Time series analysis of crude oil production in Nigeria. Adv. Multidiscip. Sci. Res. 3 (3), 1–10.

Gaspera, L. and Mbwambob, H. (2023). Forecasting Crude Oil Prices by Using ARIMA Model: Evidence from Tanzania. Journal of Accounting, Finance and Auditing Studies 9/2 (2023): 158-175. DOI: 10.32602/jafas.2023.017.

Goard, J. and AbaOud, M. A. (2023). Bimodal Model for Oil Prices. Mathematics 2023, 11, 2222. https://doi.org/10.3390/math11102222

Gong, F. and Wang, T. (2022). The Variable Volatility Elasticity Model from Commodity Markets. arXiv:2203.09177v1 [q-fin.MF] 17 Mar 2022

Gray, D. C., Osuntuyi, A. A. and Olosunde, A. A. (2021). Exponential power jump diffusion model applied to credit risk. Lap Lambert Academic Publishing, Germany.

Gunarto, T., Azhar, R., Tresiana, N., Supriyanto and Ahadiat, A. (2021). Accurate Estimated Model of Volatility Crude Oil Price. International Journal of Energy Economics and Policy, 2020, 10(5), 228-233.

Hendrawaty, E., Azhar, R., Kesumah, F. S., Sembiring, S. I. O. and Metalia, M. (2021). Modelling and Forecasting Crude Oil Prices during COVID-19 Pandemic. International Journal of Energy Economics and Policy, 2021, 11(2), 149-154. DOI: https://doi.org/10.32479/ijeep.10578.

Hughen, W. K. (2010). A maximal affine stochastic volatility model of oil prices. The Journal of Futures Markets, Vol. 30, No. 2, 101–133 (2010)

Leneenadogo, W. and Lebari, T. G. (2019). Modelling the Nigeria Crude Oil Prices Using ARIMA, Pre-intervention and Post-intervention Model. Asian Journal of Probability and Statistics, February, 1–12. https://doi.org/10.9734/ajpas/2019/v3i130083

Loera, A. L. V., Núñez Mora, J. and Aragón, M. B. (2021). Modeling Crude Oil and Refined Petroleum Product Spreads: An Alternative Tool for Risk Quantification. Nueva Época, año 29, número 54, enero-junio 2021, pp. 109-136, http://dx.doi.org/10.24275/ETYPUAM/NE/542021/Lopez

Lu, Q., Sun, S., Duan, H. and Wang, S. (2021). Analysis and forecasting of crude oil price based on the variable selection-LSTM integrated model. Energy Informatics 2021, 4(Suppl 2):47. https://doi.org/10.1186/s42162-021-00166-4

Obite, C. P., Chukwu A., Bartholomew, D. C., Nwosu, U.I. and Esiaba, G. E. (2021). Classical and machine learning modeling of crude oil production in Nigeria: Identification of an eminent model for application. Energy Reports 7 (2021) 3497–3505. https://doi.org/10.1016/j.egyr.2021.06.005

Obite, C. P., Bartholomew, D. C., Nwosu, U.I. Esiaba, G. E. and Kiwu, L. C. (2021). The optimal machine learning modeling of Brent crude oil price. Quarterly Journal of Econometrics Research 2021 Vol. 7, No. 1, pp. 31-43. DOI: 10.18488/journal.88.2021.71.31.43

Ogbogbo, C. P. (2018). Jump-Diffusion Model for Crude Oil Spot Price Process: Parameter Estimation for Predicting the Market. Ghana J. Sci. 58 (2018), 59-69

Ogbogbo, C. P. (2019). Stochastic Model of Crude Oil Spot Price Process as a Jump-Diffusion Process. Appl. Math. Inf. Sci. 13, No. 6, 1029-1037 (2019).

Øksendal, B. and Sulem A. (2019). Applied Stochastic Control of Jump Diffusions. Springer Nature, ed 3.

Onyeka-Ubaka, J. N. and Okafor, R. O. (2018). Modelling crude oil price time-varying volatility using jump-diffusion model. Unilag Journal of Medicine, Science and Technology (UJMST) Vol 6. No. 2, 2018.

Omekara, C.O., Okereke, O.E., Ire, K.I., Okamgba, C.O., 2015. ARIMA modeling of Nigeria crude oil production. J. Energy Technol. Policy 5 (10), 1–5.

Matsuda, K. (2004). Introduction to Option Pricing with Fourier Transform: Option Pricing with Exponential Lévy Models. Working Paper, Graduate School and University Center of the City University of New York.

Matsuda, H. (2011). On Quasi-Likelihood Analyses for Stochastic Differential Equations with Jumps. Int. Statistical Inst.: Proc. 58th World Statistical Congress, 2011, Dublin (Session IPS007).

Merton, R. C. (1976). Option Pricing When Underlying Stock Returns Are Continuous. Journal of Financial Economics, 3, 125-144.

Monday, T. E. and Abdulkadir, A. (2020). Modeling Fluctuation of the Price of Crude Oil in Nigeria Using ARCH, ARCH-M Models. Asian Journal of Probability and Statistics, 7(1): 16-40, 2020

Mwanakatwe, P.K., Daniel, J. and Nzungu, K.R. (2023) Forecasting Crude Oil Price Volatility by Heston Model. Journal of Mathematical Finance, 13, 408-420. https://doi.org/10.4236/jmf.2023.133026

Nairametrics, (2024). Nigeria's oil sells for about $92 a barrel, $14 higher than FG budget benchmark. Retrieved on 05 August 2024 from https://nairametrics.com/2024/04/03/nigerias-oil-sells-at-about-92-a-barrel-14-higher-than-fg-budget-benchmark/

NBS, (2024). Consumer Price Index June 2024. National Bureau of Statistics July 2024. https://nigerianstat.gov.ng/download/1241533

Ng'ang'a, F. W. and Oleche, M. (2022). Modelling and Forecasting of Crude Oil Price Volatility Comparative Analysis of Volatility Models. Journal of Financial Risk Management, 11, 154-187. https://doi.org/10.4236/jfrm.2022.111008

Nunno, G.D., Øksendal B. and Proske F. (2009). Malliavin Calculus for Lévy Processes with Applications to Finance. Springer Berlin, Heidelberg. ISBN 978-3-540-78571-2

Nwafor, C. N. and Oyedele, A. A. (2017). Simulation and hedging oil price with geometric Brownian motion and single-step binomial price model. European Journal of Business and Management, vol. 9, no. 9, pp. 68-81.

Ogihara, T. and Yoshida, N. (2011). Quasi-likelihood analysis for the stochastic differential equation with jumps. Statistical Inference for Stochastic Processes, 14(3), 189.

Rodhan, A. A. and Jaaz, A. H. (2021). Box-Jenkins Modelling and Forecasting of WTI Crude Oil Price. IMDC-IST 2021, September 07-09, Sakarya, Turkey

Shimizu, Y. and Yoshida, N. (2006). Estimation of parameters for diffusion processes with jumps from discrete observations. Statistical Inference for Stochastic Processes, 9(3), 227–277.

Suleiman, M., Muhammad, I., Adamu, A. Z., Zakari, Y., Iliyasu, R., Muhammad, A., Adamu, I. and Abdu, M. (2023). Modelling Nigeria Crude Oil Prices using ARIMA Time Series Models. NIPES Journal of Science and Technology Research, 5(1) 2023 pp. 230-241

Usoro A.E., Ekong A. (2022), Modeling Nigeria Crude Oil Production and Price Volatility Using Multivariate Generalized Autoregressive Conditional Heteroscedasticity Models. African Journal of Mathematics and Statistics Studies 5(1), 33-54. DOI: 10.52589/AJMSS-L4FI9DW6

Uchida, M. and Yoshida, N. (2012). Nondegeneracy of Random Field and Estimation of Diffusion.arXiv:1212.5715 [math.ST], URL http://arxiv.org/abs/1212.5715.

Vanguard News (2024). Oil Market: Crude oil price rises to $88.97 per barrel. Retrieved 05 August 2024 from https://www.vanguardngr.com/2024/04/oil-market-crude-oil-price-rises-to-88-97-per-barrel/