A NEW FRÉCHET-G FAMILY OF CONTINUOUS PROBABILITY DISTRIBUTIONS: SPECIAL MODELS, PROPERTIES, SIMULATION AND APPLICATIONS

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Abstract:
This article derived a new family of continuous probability distributions known as New Frèchet-G family of distributions. The study presents expansion and linear representation of the CDF and CDF of the new Frèchet-G (NFr-G) family of distribution. The structural properties of the proposed family of distribution, such as quantile function, moment, moment generating function, characteristic function, reliability analysis and order statistics were derived. The parameters of the new family have been estimated using the method of maximum likelihood estimation. Four special models have been derived and discussed from the new Frèchet-G family. The shapes of their densities and CDFs of the special models for some chosen parameter values were also discussed. The plots of the PDFs of the special models revealed that the NFr-G family will generate distributions with different shapes which is an indication that the proposed family is skewed and flexible. Monte Carlo simulation study was conducted to check the stability of the parameters of the proposed family through the NFrExD and the results revealed that the estimators of the parameters are consistent. The performance of the proposed family was assessed via the NFrExD by some applications of the model to real life data sets.

Key words: Frèchet distribution, Frèchet-G family, Frèchet-exponential distribution, Maximum Likelihood Estimation, Simulation, Real life data.

1. Introduction

The functions, properties and interrelationships of probability distributions are very important in describing real life events. Many standard probability models have been used in the past decades for modeling data in several fields. However, extending these classical distributions has produced several compound or modified distributions that are found to be more flexible and appropriate for modelling events as compared to their standard forms. Sequel to this fact, a large number of useful families of distributions have been proposed in literature such as quadratic rank transmutation map by Shaw and Buckley (2007), Exponentiated T-X by Alzaghal *et al*. (2013), Weibull-X by Alzaatreh *et al*. (2013), Weibull-G by Bourguignon *et al.* (2014), a Lomax-G family by Cordeiro *et al.* (2014), a new Weibull-G family by Tahir *et al.* (2016), a Lindley-G family by Cakmakyapan and Ozel (2016), a Gompertz-G family by Alizadeh *et al.* (2017) and Odd Lindley-G family by Gomes-Silva *et al*. (2017), odd Lomax-G family by Cordeiro *et al.*

(2019), the shifted Gompertz-G family of distributions by Eghwerido *et al*. (2021a), the shifted exponential-G family of distributions by Eghwerido *et al*. (2022), the transmuted alpha power-G family of distributions by Eghwerido *et al*. (2021b), the alpha power Marshall-Olkin-G distribution by Eghwerido *et al*. (2021c), a new flexible odd Kappa-G family of distributions by Al-Shomrani and Al-Arfaj (2021), an odd Chen-G family of distributions by Anzagra *et al*. (2022), a new sine family of generalized distributions by Benchiha *et al*. (2023), the Topp-Leone type II exponentiated half logistic-G family of distributions by Gabanakgosi and Oluyede (2023), a truncated Cauchy power Weibull-G class of distributions by Alotaibi *et al*. (2022), the truncated Burr X-G family of distributions by Bantan *et al*. (2021), the flexible Burr X-G family of distribution by Al-Babtain *et al*. (2021), the Marshall-Olkin-odd power generalized Weibull-G family of distributions by Chipepa *et al*. (2022), a novel bivariate Lomax-G family of distributions Fayomi *et al*. (2023) and X-exponential-G Family of Distributions by Mohammad (2024).

The Fréchet distribution is a very useful distribution especially in extreme value theory and it is applied in different areas such as accelerated life testing as well as earthquakes, floods, horse racing, rainfall, queues in supermarkets, wind speeds and sea waves etc. Details of the properties, importance and applications of the Fréchet distribution can be obtained from Kotz and Nadarajah (2000), Harlow (2002), Nadarajah and Kotz (2008), Zaharim *et al.* (2009) and Mubarak (2012). A random variable *T* is said to follow a Fréchet distribution with parameters a and b if its probability density function (PDF) and cumulative distribution function (CDF) are respectively given by:

$$f\left(t\right)=ab^{a}t^{-a-1}\mathrm{e}^{-\left(\frac{b}{t}\right)^{a}} \qquad (1)$$

and

$$F\left(t\right)=\mathrm{e}^{-\left(\frac{b}{t}\right)^{a}} \qquad (2)$$

where $t>0,a>0,b>0$; while a and b are the shape and scale parameters of the Fréchet distribution respectively.

Using the Frèchet distribution above, many researchers have developed Frèchet based families of probability distributions which include, the Odd Frèchet Generalised (OFr-G) family of distributions by Haq and Elgarhy (2018), the extended odd Fréchet (EOFr-G) family of distributions by Nasiru (2018), the Fréchet Topp Leone-G (FTL-G) family by Reyad *et al*. (2019), the Generalized Odd Frèchet-G family of distributions by Marganpoor *et al*., (2020) and the exponentiated Frèchet-G family of distribution by Baharith and Alamoudi (2021).

Based on the literature reviewed, it is discovered that there are very scanty distributions that are based on the Fréchet-G family unlike other families of distributions and hence the need for more Frèchet-based distributions. It is also discovered that the odd Frèchet-G family (with link function, $G\left(x\right)/1-G\left(x\right)$) by Haq and Elgarhy (2018) and the extended odd Frèchet-G family (with link function, $G\left(x\right)^{\alpha}/1-G\left(x\right)^{\alpha}$) by Nasiru (2018) do not include the scale parameter of the Frèchet distribution which is supposed to add to the shape of the Fréchet families. Also, it has been discovered that no Frèchet-based family has considered the link function, $-\log\left[1-G(x)\right]$ as proposed by Alzaatreh *et al*., (2013a), which has been found to be efficient in other families of distributions (Cordeiro *et al*., (2014); Alzaatreh *et al*., (2013b), Cakmakyapan and Ozel (2016); Alizadeh *et al*., (2017)).

Therefore, this article introduced and studied a new Fréchet-G family (NFr-G) by employing the Transformed–Transformer (T-X) method proposed by Alzaatreh *et al*., (2013a) with the link function, $-\log\left[1-G(x)\right]$. The proposed NFr-G family is found to be flexible with tractable sub-models and will aid the generalization of many other continuous distributions with various shapes and a wide range of applications.

The remaining sections of this article are presented as follows: The NFr-G family and reliability functions are defined in section 2. Some special distributions from the NFr-G family are derived in section 3 and the expansion and linear representation of the CDF and PDF of the NFr-G family is given in section 4. In section 5, the article derived the structural properties of the NFr-G family. The estimation of parameters of the NFr-G family and a simulation study to check the consistency of the estimators is done in section 6 and 7 respectively. The performance of the NFr-G family is illustrated using three real life datasets with respect to one of the special models in section 8. The conclusion of the study is presented in section 9.

2. New Frechet-G Family of Continuous Probability Distributions

Considering the method of generating families of continuous probability distributions (the Transformed-Transformer, "*T-X*" method) by Alzaatreh *et al*. (2013a), the CDF and PDF of the proposed new Frechet-G family of probability distributions are defined respectively (for $x>0$) as:

$$F(x;a,b,\eta)=\int_0^{W\left[G(x;\eta)\right]} f(t)dt=\int_0^{-\log\left[1-G(x;\eta)\right]} ab^a t^{-a-1}\mathrm{e}^{-\left(\frac{b}{t}\right)^a}dt=\exp\left\{-b^a\left(-\log\left[1-G(x;\eta)\right]\right)^{-a}\right\} \quad (3)$$

and

$$f(x;a,b,\eta)=\frac{ab^a g(x;\eta)\left(-\log\left[1-G(x;\eta)\right]\right)^{-a-1}}{\left[1-G(x;\eta)\right]}\exp\left\{-b^a\left(-\log\left[1-G(x;\eta)\right]\right)^{-a}\right\} \quad (4)$$

where $a>0$ and $b>0$ are the two shape parameters that will enhance the flexibility and shape of the new family, $g(x)$ and $G(x)$ are the PDF and CDF of any baseline distribution respectively and η is a $p\times 1$ vector of parameters for the baseline $G(x;\eta)$.

Considering the results in equation (3) and (4), the survival and hazard functions (reliability functions) of the new Frechet-G family of distribution are defined respectively as:

$$S(x,a,b,\eta)=1-F(x,a,b,\eta)=1-\exp\left\{-b^a\left(-\log\left[1-G(x;\eta)\right]\right)^{-a}\right\} \quad (5)$$

and

$$h(x;a,b,\eta)=\frac{f(x,a,b,\eta)}{S(x,a,b,\eta)}=\frac{ab^a g(x;\eta)\left(-\log\left[1-G(x;\eta)\right]\right)^{-a-1}\exp\left\{-b^a\left(-\log\left[1-G(x;\eta)\right]\right)^{-a}\right\}}{\left[1-G(x;\eta)\right]\left(1-\exp\left\{-b^a\left(-\log\left[1-G(x;\eta)\right]\right)^{-a}\right\}\right)} \quad (6)$$

3. Special NFr Distributions

This section derives some special distributions generated from the new family to show its usefulness for extending every other continuous probability distribution, with some important plots using arbitrary values of their parameters.

3.1 Fréchet-exponential distribution

The PDF and CDF of the exponential distribution are defined respectively as $g(x)=\lambda e^{-\lambda x}$ and $G(x)=1-e^{-\lambda x}$, where for $x>0$ and $\lambda>0$ is the scale parameter of the distribution. Substituting the pdf and cdf of the exponential distribution into the NFr-G family of distribution in equation (3) and (4) accordingly and simplifying, the cdf and pdf of the New Frechet-exponential distribution (NFrExD) are obtained as given in Equation (7) and (8) respectively below:

$$F(x)=\exp\left\{-\left(\frac{b}{\lambda}\right)^a x^{-a}\right\} \quad (7)$$

and

$$f(x)=a\left(\frac{b}{\lambda}\right)^a x^{-a-1}\exp\left\{-\left(\frac{b}{\lambda}\right)^a x^{-a}\right\} \quad (8)$$

where $a,b,\lambda>0,$ are the parameters of the NFrExD.

3.2 Fréchet-Weibull distribution

The PDF and CDF of the Lindley distribution are defined respectively as $g(x)=\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$ and $G(x)=1-e^{-\alpha x^{\beta}}$, where $x>0,\alpha>0,\beta>0$, α *and* β are the scale and shape parameters respectively. Substituting the pdf and cdf of the Weibull distribution into the NFr-G family in Equation (3) and (4) accordingly and simplifying, the cdf and pdf of the new Frechet-Weibull distribution (NFrWeiD) are obtained as given in Equation (9) and (10) respectively:

$$F(x)=\exp\left\{-\left(\frac{b}{\alpha}\right)^a x^{-a\beta}\right\} \quad (9)$$

and

$$f(x)=a\beta\left(\frac{b}{\alpha}\right)^a x^{-a\beta-1}\exp\left\{-\left(\frac{b}{\alpha}\right)^a x^{-a\beta}\right\} \quad (10)$$

where $a,b,\alpha,\beta>0$ are the parameters of the NFrWeiD.

3.3 Fréchet-Gompertz distribution

The PDF and CDF of the Gompertz distribution with parameters α and β are defined respectively as $g(x)=\alpha e^{\beta x} e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}$ and $G(x)=1-e^{-\frac{\alpha}{\beta}\left(e^{\beta x}-1\right)}$, where $x \geq 0, \alpha>0, \beta>0$, α is a scale parameter and β is a shape parameter of the Gompertz distribution. Substituting the pdf and cdf of the Gompertz distribution into the NFr-G family in Equation (3) and (4) accordingly and simplifying, the cdf and pdf of the new Frechet-Gompertz distribution (NFrGomD) are obtained as given in Equation (11) and (12) respectively:

$$F(x)=\exp\left\{-\left(\frac{b\beta}{\alpha}\right)^{a}\left(e^{\beta x}-1\right)^{-a}\right\} \tag{11}$$

and

$$f(x)=a\beta\left(\frac{b\beta}{\alpha}\right)^{a} e^{\beta x}\left(e^{\beta x}-1\right)^{-a-1}\exp\left\{-\left(\frac{b\beta}{\alpha}\right)^{a}\left(e^{\beta x}-1\right)^{-a}\right\} \tag{12}$$

where $a, b, \alpha, \beta>0,$ are the parameters of the NFrGomD.

3.4 Fréchet-Dagum distribution

The PDF and CDF of the Dagum distribution are defined respectively as $g(x)=\alpha\theta\beta x^{-\theta-1}\left(1+\alpha x^{-\theta}\right)^{-\beta-1}$ and $G(x)=\left(1+\alpha x^{-\theta}\right)^{-\beta}$, where $x>0$, $\alpha, \theta, \beta>0$. The scale parameter is α while θ and β are the shape parameters. Substituting the pdf and cdf of the Dagum distribution into the NFr-G family in Equation (3) and (4) accordingly and simplifying, the cdf and pdf of the new Frechet-Dagum distribution (NFrDaD) are obtained as given in Equation (13) and (14) respectively:

$$F(x)=\exp\left\{-b^{a}\left(-\log\left[1-\left(1+\alpha x^{-\theta}\right)^{-\beta}\right]\right)^{-a}\right\} \tag{13}$$

and

$$f(x)=\frac{ab^{a}\alpha\theta\beta x^{-\theta-1}\left(-\log\left[1-\left(1+\alpha x^{-\theta}\right)^{-\beta}\right]\right)^{-a-1}}{\left(1+\alpha x^{-\theta}\right)^{\beta+1}\left[1-\left(1+\alpha x^{-\theta}\right)^{-\beta}\right]}\exp\left\{-b^{a}\left(-\log\left[1-\left(1+\alpha x^{-\theta}\right)^{-\beta}\right]\right)^{-a}\right\} \tag{14}$$

where $a, b, \alpha, \beta, \theta>0,$ are the parameters of the NFrDaD.

The Figures 1(a)-1(d) present the PDF and CDF of some special distributions using arbitrary parameter values.

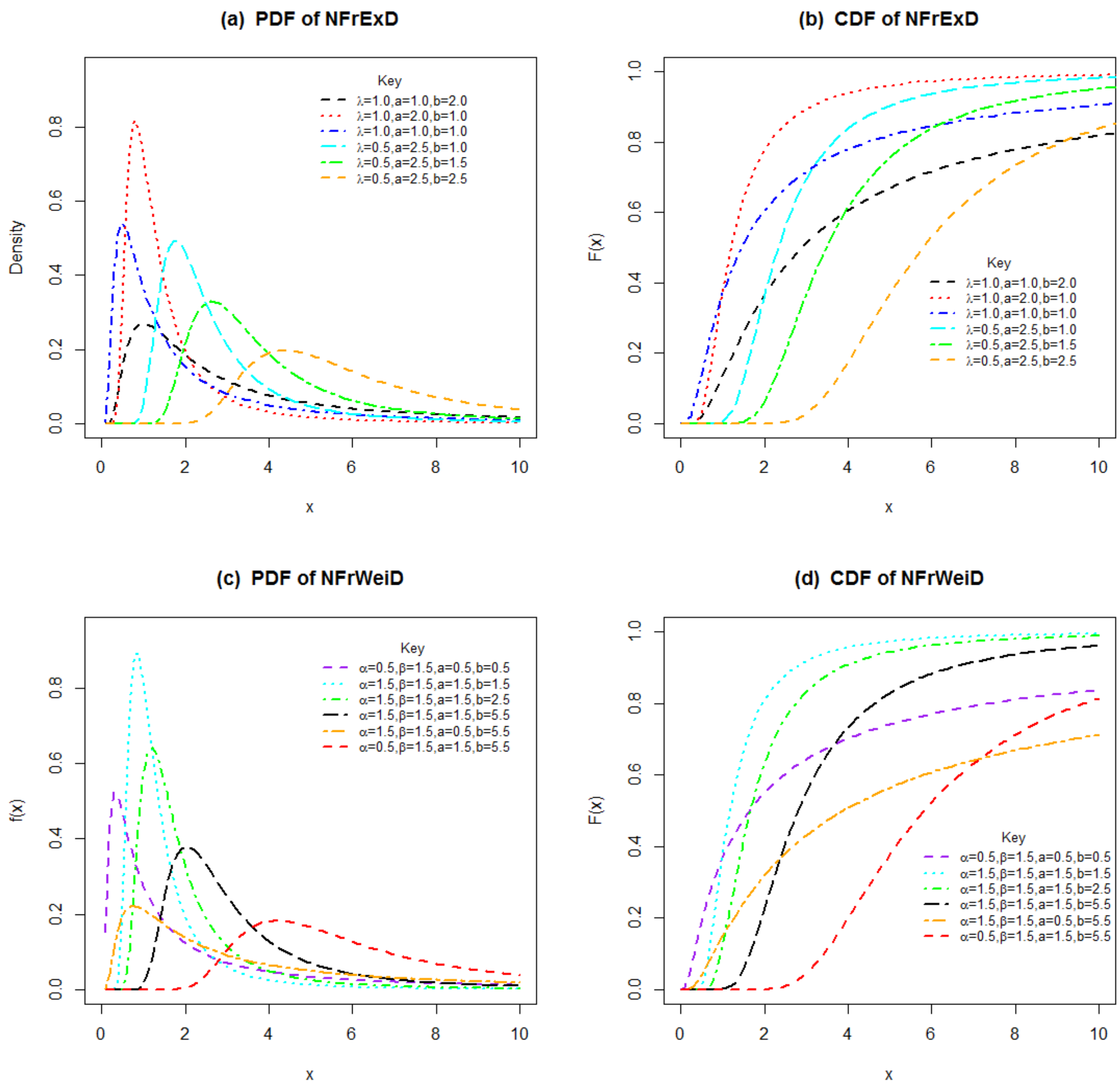

Figure 1: PDF and CDF of NFrExD and NFrWeiD for selected parameter values.

The plots of the PDFs of the NFrExD and NFrWeiD show that the proposed family of distribution is skewed and flexible and that its shape depends on the values of the parameters. Also, the shapes can be classified as decreasing, unimodal, right-skewed and asymmetrical shapes, as shown in Figures 1(a) and 1(c). The plot of CDFs all converges to one as expected which also confirms the validity of the NFr-G family of probability distribution as displayed in Figures 1(b) and 1(d) for NFrExD and NFrWeiD respectively.

4. Useful Expansions of the CDF and PDF of the NFr-G Family of Distribution.

This section presents a useful expansion and mixture or linear representation of the CDF and PDF of the new Frechet-G family of distribution.

Recall that according to Taylor series expansion, $\exp(-z)=\sum_{i=0}^{\infty}\frac{(-1)^i z^i}{i!}$, this implies that the exponential term in the CDF and PDF of the NFr-G family can be expressed as:

$$\exp\left\{-b^a\left(-\log\left[1-G(x)\right]\right)^{-a}\right\}=\sum_{i=0}^{\infty}\frac{(-1)^i b^{ai}}{i!}\left(-\log\left[1-G(x)\right]\right)^{-ai} \quad (15)$$

Also, according to Tahir *et al*., (2016), the logaritmic function in Equation (15) can be expressed as:

$$\left(-\log\left[1-G(x)\right]\right)^{-ai}=\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k+l}(ai)}{(ai-j)}\binom{k-ai}{k}\binom{k}{j}\binom{ai+k}{l}P_{j,k}\left[1-G(x)\right]^l \quad (16)$$

where for (for $j\geq 0$), $P_{j,0}=1$ and (for $k=1,2,3,\ldots$), $P_{j,k}=k^{-1}\sum_{m=1}^{k}(-1)^m\frac{\left[m(j+1)-k\right]}{(m+1)}P_{j,k-m}$

Now, for l greater than zero, we can expand the last term in Equation (16) as:

$$\left[1-G(x)\right]^l=\sum_{m=0}^{\infty}(-1)^m\binom{l}{m}\left[G(x)\right]^m \quad (17)$$

Using Equation (16) and (17) in Equation (15) and simplifying, we have:

$$\exp\left\{-b^a\left(-\log\left[1-G(x)\right]\right)^{-a}\right\}=\sum_{i,k,l,m=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l+m}(ai)}{i!(ai-j)b^{-ai}}\binom{k-ai}{k}\binom{k}{j}\binom{ai+k}{l}\binom{l}{m}P_{j,k}\left[G(x)\right]^m$$

Therefore, the mixture form expression of the CDF of the NFr-G family, $F(x)$, becomes:

$$F(x)=\sum_{i,k,l,m=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l+m}(ai)}{i!(ai-j)b^{-ai}}\binom{k-ai}{k}\binom{k}{j}\binom{ai+k}{l}\binom{l}{m}P_{j,k}\left[G(x)\right]^m$$

Hence,

$$F(x)=\sum_{l=0}^{\infty}\psi_l\left[G(x)\right]^l \quad (18)$$

where $\psi_l=\sum_{k,l,m=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l+m}(ai)}{i!(ai-j)b^{-ai}}\binom{k-ai}{k}\binom{k}{j}\binom{ai+k}{l}\binom{l}{m}P_{j,k}$ is a constant.

Similarly, using the expansion in Equation (15), (16) and (17), the mixture form expression of the PDF of the NFr-G family, $f(x;a,b,\eta)$, is given as:

$$f\left(x;a,b,\eta\right)=\sum_{i=0}^{\infty}\psi_{l+1}g\left(x;\eta\right)\left[G\left(x\right)\right]^{m} \tag{19}$$

where

$$\psi_{l+1}=\sum_{k,l,m=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l+m}\,a\left(a\left(i+1\right)+1\right)}{b^{-a(i+1)}\left(\left(a\left(i+1\right)+1\right)-j\right)i!}\binom{k-\left(a\left(i+1\right)+1\right)}{k}\binom{k}{j}\binom{\left(a\left(i+1\right)+1\right)+k}{l}\binom{l-1}{m}P_{j,k}$$

is a constant.

5. Mathematical Properties of the New Frechet-G Family of Distribution

This section presents various structural, statistical, and mathematical properties of the NFr-G family of probability distribution.

5.1 Quantile Function

Using the CDF of the NFr-G family defined in Equation (3), the quantile function is obtained as follows;

$$F(x,a,b,\eta)=\exp\left\{-b^{a}\left(-\log\left[1-G\left(x;\eta\right)\right]\right)^{-a}\right\}=u \tag{20}$$

Simplifying the expression above, the quantile function of the NFr-G family is given by;

$$Q\left(u\right)=Q_{G}\left(\left(1-\exp\left\{-\left(-b^{-a}\ln\left(u\right)\right)^{-\frac{1}{a}}\right\}\right);\eta\right) \tag{21}$$

where $Q_{G}\left(.;.\right)=G^{-1}\left(.;.\right)$ is the baseline quantile function that correspond to the CDF of the baseline distribution, $G\left(x;\eta\right)$.

5.2 Moments

The r^{th} non-central moment of a random variable X is defined as;

$$\mu_{r}^{'}=E\left[X^{r}\right]=\int_{0}x^{r}f\left(x\right)dx \tag{22}$$

Substituting the mixture representation of the density function into the definition above simplifying produces:

$$\mu_{r}^{'}=\sum_{i=0}^{\infty}\psi_{l+1}\int_{0}^{\infty}x^{r}g\left(x;\eta\right)\left[G\left(x\right)\right]^{m}dx \tag{23}$$

5.3 Moment Generating Function

The moment generating function of a random variable X can be obtained as;

$$M_{X}\left(t\right)=E\left[e^{tx}\right]=\int_{0}^{\infty}e^{tx}f\left(x\right)dx \tag{24}$$

Using power series expansion and simplifying, the moment generating function of a random variable X can also be expressed as:

$$M_X(t)=\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\frac{t^r}{r!}\psi_{l+1}\int_0^{\infty}x^r g(x;\eta)\left[G(x)\right]^m dx \tag{25}$$

5.4 Characteristics Function

A representation for the characteristics function is given by:

$$\phi_x(t)=E\left(e^{itx}\right)=\int_0^{\infty}e^{itx}f(x)dx \tag{26}$$

Hence, simple algebra and use of the power series expansion gives the following results:

$$\phi_X(t)=\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\frac{(it)^r}{r!}\psi_{l+1}\int_0^{\infty}x^r g(x;\eta)\left[G(x)\right]^m dx \tag{27}$$

5.5 Distribution of Order Statistics

Suppose $X_1, X_2, ..., X_n$ is a random sample from a distribution with PDF, $f(x)$, and let $X_{1:n}, X_{2:n}, ..., X_{i:n}$ denote the corresponding order statistic obtained from this sample. The PDF, $f_{i:n}(x)$ of the i^{th} order statistic can be defined as;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{\infty}(-1)^k\binom{n-i}{k}f(x)\left[F(x)\right]^{k+i-1} \tag{28}$$

Taking $f(x)$ and $F(x)$ to be the PDF and CDF of the NFr-G family respectively and using Equation (3) and (4), the PDF of the i^{th} order statistics $X_{i:n}$ for the NFr-G family can be expressed from (28) as;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{k=0}^{\infty}(-1)^k\binom{n-i}{k}\frac{ag(x;\eta)\exp\left\{-b^a\left(-\log\left[1-G(x;\eta)\right]\right)^{-a}\right\}}{b^{-a}\left[1-G(x;\eta)\right]\left(-\log\left[1-G(x;\eta)\right]\right)^{a+1}}\left[\exp\left\{-b^a\left(-\log\left[1-G(x;\eta)\right]\right)^{-a}\right\}\right]^{k+i-1}$$

Hence, the PDF of the minimum order statistic $X_{(1)}$ and maximum order statistic $X_{(n)}$ of the NFr-G family are given by;

$$f_{1:n}(x)=n\sum_{k=0}^{\infty}(-1)^k\binom{n-1}{k}\frac{ag(x;\eta)\exp\left\{-b^a\left(-\log\left[1-G(x;\eta)\right]\right)^{-a}\right\}}{b^{-a}\left[1-G(x;\eta)\right]\left(-\log\left[1-G(x;\eta)\right]\right)^{a+1}}\left[\exp\left\{-b^a\left(-\log\left[1-G(x;\eta)\right]\right)^{-a}\right\}\right]^k \tag{29}$$

and

$$f_{n:n}(x)=n\frac{ag(x;\eta)\exp\left\{-b^{a}\left(-\log\left[1-G(x;\eta)\right]\right)^{-a}\right\}}{b^{-a}\left[1-G(x;\eta)\right]\left(-\log\left[1-G(x;\eta)\right]\right)^{a+1}}\left[\exp\left\{-b^{a}\left(-\log\left[1-G(x;\eta)\right]\right)^{-a}\right\}\right]^{n-1} \quad (30)$$

respectively.

6. Estimation of Parameters of the NFr-G Family

In this section, we used the maximum likelihood estimation (MLE) method to derive estimators for the unknown parameters of the NFr-G family of distribution. Let $X_1, X_2, ..., X_n$ be a sample of size 'n' independently and identically distributed random variables from the NFr-G family with unknown parameters, a, b and η defined previously. The likelihood function is given by:

$$L(X \mid a,b,\eta)=\left(ab^{a}\right)^{n}\prod_{i=1}^{n}\left(\frac{g(x_i;\eta)\left(-\log\left[1-G(x_i;\eta)\right]\right)^{-a-1}}{\left[1-G(x_i;\eta)\right]}\right)\exp\left\{-b^{a}\sum_{i=1}^{n}\left(-\log\left[1-G(x_i;\eta)\right]\right)^{-a}\right\} \quad (31)$$

For ease of differentiation, we take the natural logarithm of the likelihood function in Equation (31) and let the log-likelihood function be $l=\log L(X \mid a,b,\eta)$, such that:

$$l=n\log a+an\log b+\sum_{i=1}^{n}\log\left\{g(x_i;\eta)\right\}-(a+1)\sum_{i=1}^{n}\log\left(-\log\left[1-G(x_i;\eta)\right]\right)-\sum_{i=1}^{n}\log\left[1-G(x_i;\eta)\right]-b^{a}\sum_{i=1}^{n}\left(-\log\left[1-G(x_i;\eta)\right]\right)^{-a}$$

(32)

Differentiating l in Equation (37) partially with respect to the parameters, a, b and η respectively gives;

$$\frac{\partial l}{\partial a}=\frac{n}{a}+n\log b-\sum_{i=1}^{n}\log\left(\log\left[G(x_i;\eta)\right]\right)-b^{a}\sum_{i=1}^{n}\left(\log\left[G(x_i;\eta)\right]\right)^{-a}\left\{\ln b-\ln\left(\log\left[G(x_i;\eta)\right]\right)\right\} \quad (33)$$

$$\frac{\partial l}{\partial b}=\frac{an}{b}-ab^{a-1}\sum_{i=1}^{n}\left(-\log\left[1-G(x_i;\eta)\right]\right)^{-a} \quad (34)$$

$$\frac{\partial l}{\partial \eta}=\sum_{i=1}^{n}\left[\frac{g'(x_i;\eta)}{g(x_i;\eta)}\right]-(a+1)\sum_{i=1}^{n}\left[\frac{G'(x_i;\eta)}{\left(\log\left[G(x_i;\eta)\right]\right)G(x_i;\eta)}\right]-\sum_{i=1}^{n}\left[\frac{G'(x_i;\eta)}{G(x_i;\eta)}\right]-ab^{a}\sum_{i=1}^{n}\left[\frac{\left(\log\left[G(x_i;\eta)\right]\right)^{-a-1}G'(x_i;\eta)}{G(x_i;\eta)}\right]$$

(35)

where $g'(x;\eta)=\frac{\partial g(x;\eta)}{\partial \eta}$ and $G'(x;\eta)=\frac{\partial G(x;\eta)}{\partial \eta}$. The solution of the non-linear system of Equations $\frac{\partial l}{\partial a}$, $\frac{\partial l}{\partial b}$ and $\frac{\partial l}{\partial \eta}$ will give us the maximum likelihood estimates of parameters, a, b and η. However, the solution cannot be obtained analytically except numerically with the aid of suitable statistical software like Python, R, SAS, Mapple, e.t.c when data sets are available.

7. Simulation study

This section assesses the behavior of the maximum likelihood estimators (MLEs) of the NFrExD for a finite sample of size n. A simulation study based on the NFrExD is performed. The generation of random numbers is done by the quantile technique from the NFrExD using optim() R-function with the argument method "L-BFGS-B". The simulation study is based on the following steps:

(1) Generation of N=1000 samples of sizes n =25, 50, . . . , 1000 from the NFrExD

(2) Computation of maximum likelihood estimates for the model parameters

(3) Computation of the mean square errors (MSEs) and biases (absolute biases) given by

$\mathrm{MSE}(\Theta)=\frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\Theta}-\Theta\right)^2$ and $\mathrm{Bias}(\Theta)=\frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\Theta}-\Theta\right)$ respectively, where $\Theta=(a,b,\lambda)$ and $\hat{\Theta}=\left(\hat{a},\hat{b},\hat{\lambda}\right)$.

The simulation study is conducted for four different combination of λ, a and b. These are:
(i) $\lambda=1.0,\ a=1.0,\ and\ b=1.0$ (ii) $\lambda=2.0,\ a=1.0,\ and\ b=1.0$
(iii) $\lambda=1.0,\ a=1.0,\ and\ b=2.0$ and (iv) $\lambda=0.5,\ a=1.0,\ and\ b=0.5$.

The assessment of the estimators, $\hat{\lambda}_{MLE},\ \hat{a}_{MLE},\ and\ \hat{b}_{MLE}$ is done by using MSE and Bias. For every sample size, the average MLEs, MSEs, Absolute biases were computed. The results obtained are presented in Tables 1-4 and displayed graphically in Figures 2–5 as follows:

Table 1: Simulation results of the NFrExD when $\lambda=1.0,\ a=1.0,\ and\ b=1.0$

n	Measures/Criteria	Parameters			n	Measures/Criteria	Parameters		
		λ	a	b			λ	a	b
n=25	MLEs	1.0500	1.0600	1.0708	**n=300**	MLEs	1.0160	1.0044	1.0146
	Biases	0.0500	0.0600	0.0708		Biases	0.0160	0.0044	0.0146
	MSEs	0.0182	0.0328	0.0270		MSEs	0.0014	0.0021	0.0012
n=50	MLEs	1.0371	1.0283	1.0430	**n=500**	MLEs	1.0100	1.0009	1.0135
	Biases	0.0371	0.0283	0.0430		Biases	0.0100	0.0009	0.0135
	MSEs	0.0088	0.0150	0.0100		MSEs	0.0007	0.0012	0.0008
n=75	MLEs	1.0296	1.0163	1.0352	**n=700**	MLEs	1.0106	1.0018	1.0093
	Biases	0.0296	0.0163	0.0352		Biases	0.0106	0.0018	0.0093
	MSEs	0.0055	0.0093	0.0064		MSEs	0.0006	0.0008	0.0005
n=100	MLEs	1.0243	1.0119	1.0283	**n=900**	MLEs	1.0084	1.0008	1.0090
	Biases	0.0243	0.0119	0.0283		Biases	0.0084	0.0008	0.0090
	MSEs	0.0039	0.0061	0.0041		MSEs	0.0004	0.0007	0.0004
n=200	MLEs	1.0202	1.0107	1.0183	**n=1000**	MLEs	1.0072	1.0014	1.0095
	Biases	0.0202	0.0107	0.0183		Biases	0.0072	0.0014	0.0095
	MSEs	0.0022	0.0034	0.0020		MSEs	0.0004	0.0006	0.0004

Table 2: Simulation results of the NFrExD when $\lambda = 2.0,\ a = 1.0,\ and\ b = 1.0$

n	Measures/Criteria	Parameters			n	Measures/Criteria	Parameters		
		λ	a	b			λ	a	b
n=25	MLEs	2.0363	1.0687	1.0391	**n=300**	MLEs	2.0180	1.0040	1.0060
	Biases	0.0363	0.0687	0.0391		Biases	0.0180	0.0040	0.0060
	MSEs	0.0246	0.0426	0.0249		MSEs	0.0024	0.0021	0.0016
n=50	MLEs	2.0253	1.0274	1.0299	**n=500**	MLEs	2.0110	1.0023	1.0075
	Biases	0.0253	0.0274	0.0299		Biases	0.0110	0.0023	0.0075
	MSEs	0.0120	0.0139	0.0117		MSEs	0.0013	0.0011	0.0010
n=75	MLEs	2.0282	1.0141	1.0189	**n=700**	MLEs	2.0112	1.0029	1.0046
	Biases	0.0282	0.0141	0.0189		Biases	0.0112	0.0029	0.0046
	MSEs	0.0086	0.0087	0.0076		MSEs	0.0010	0.0009	0.0007
n=100	MLEs	2.0206	1.0163	1.0195	**n=900**	MLEs	2.0091	1.0018	1.0050
	Biases	0.0206	0.0163	0.0195		Biases	0.0091	0.0018	0.0050
	MSEs	0.0066	0.0069	0.0060		MSEs	0.0007	0.0007	0.0006
n=200	MLEs	2.0157	1.0084	1.0126	**n=1000**	MLEs	2.0089	1.0022	1.0041
	Biases	0.0157	0.0084	0.0126		Biases	0.0089	0.0022	0.0041
	MSEs	0.0031	0.0032	0.0028		MSEs	0.0007	0.0006	0.0005

Table 3: Simulation results of the NFrExD when $\lambda = 1.0,\ a = 1.0,\ and\ b = 2.0$

n	Measures/Criteria	Parameters			n	Measures/Criteria	Parameters		
		λ	a	b			λ	a	b
n=25	MLEs	1.0204	1.0613	2.0513	**n=300**	MLEs	1.0062	1.0026	2.0176
	Biases	0.0204	0.0613	0.0513		Biases	0.0062	0.0026	0.0176
	MSEs	0.0206	0.0369	0.0286		MSEs	0.0017	0.0019	0.0024
n=50	MLEs	1.0155	1.0283	2.0396	**n=500**	MLEs	1.0058	1.0022	2.0130
	Biases	0.0155	0.0283	0.0396		Biases	0.0058	0.0022	0.0130
	MSEs	0.0103	0.0138	0.0149		MSEs	0.0011	0.0012	0.0016
n=75	MLEs	1.0146	1.0169	2.0294	**n=700**	MLEs	1.0050	1.0021	2.0104
	Biases	0.0146	0.0169	0.0294		Biases	0.0050	0.0021	0.0104
	MSEs	0.0073	0.0082	0.0091		MSEs	0.0008	0.0009	0.0010
n=100	MLEs	1.0115	1.0127	2.0289	**n=900**	MLEs	1.0042	1.0028	2.0090
	Biases	0.0115	0.0127	0.0289		Biases	0.0042	0.0028	0.0090
	MSEs	0.0051	0.0065	0.0077		MSEs	0.0005	0.0007	0.0007
n=200	MLEs	1.0070	1.0080	2.0229	**n=1000**	MLEs	1.0038	1.0008	2.0097
	Biases	0.0070	0.0080	0.0229		Biases	0.0038	0.0008	0.0097
	MSEs	0.0028	0.0032	0.0040		MSEs	0.0005	0.0006	0.0007

Table 4: Simulation results of the NFrExD when $\lambda = 0.5,\ a = 1.0,\ and\ b = 0.5$

n	Measures/Criteria	Parameters λ	a	b	n	Measures/Criteria	Parameters λ	a	b
n=25	MLEs	0.5589	1.0515	0.5675	**n=300**	MLEs	0.5147	1.0045	0.5152
	Biases	0.0589	0.0515	0.0675		Biases	0.0147	0.0045	0.0152
	MSEs	0.0099	0.0336	0.0120		MSEs	0.0006	0.0021	0.0006
n=50	MLEs	0.5380	1.0334	0.5421	**n=500**	MLEs	0.5104	1.0045	0.5110
	Biases	0.0380	0.0334	0.0421		Biases	0.0104	0.0045	0.0110
	MSEs	0.0043	0.0149	0.0049		MSEs	0.0003	0.0012	0.0003
n=75	MLEs	0.5296	1.0144	0.5314	**n=700**	MLEs	0.5092	1.0031	0.5095
	Biases	0.0296	0.0144	0.0314		Biases	0.0092	0.0031	0.0095
	MSEs	0.0024	0.0089	0.0029		MSEs	0.0002	0.0009	0.0002
n=100	MLEs	0.5274	1.0160	0.5276	**n=900**	MLEs	0.5083	1.0003	0.5080
	Biases	0.0274	0.0160	0.0276		Biases	0.0083	0.0003	0.0080
	MSEs	0.0019	0.0069	0.0022		MSEs	0.0002	0.0007	0.0002
n=200	MLEs	0.5168	1.0084	0.5188	**n=1000**	MLEs	0.5076	1.0021	0.5074
	Biases	0.0168	0.0084	0.0188		Biases	0.0076	0.0021	0.0074
	MSEs	0.0008	0.0030	0.0009		MSEs	0.0002	0.0006	0.0001

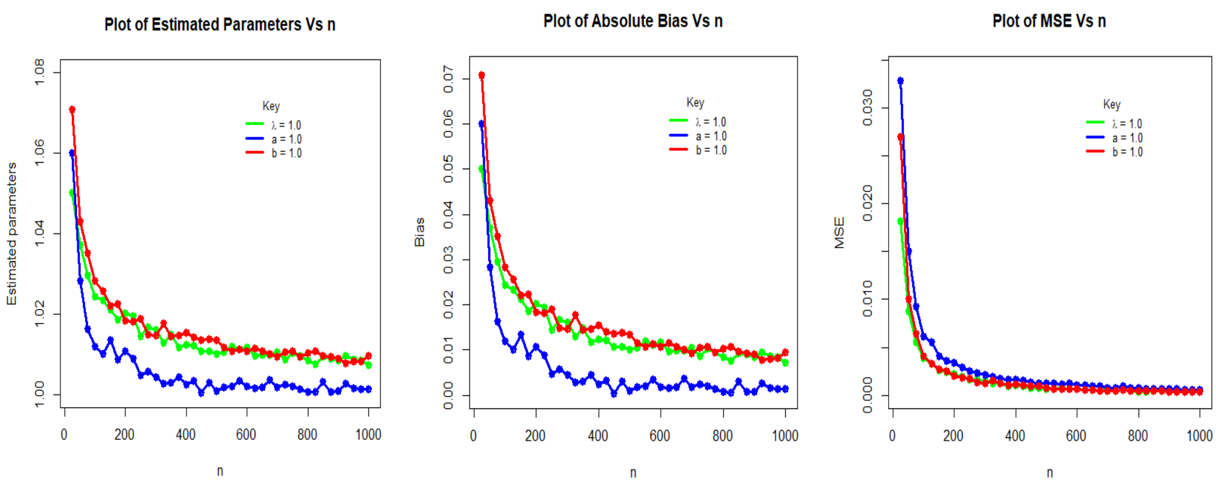

Figure 2: Plots of MLEs, Absolute Biases and MSEs of the NFrExD when $\lambda = 1.0,\ a = 1.0,\ and\ b = 1.0$

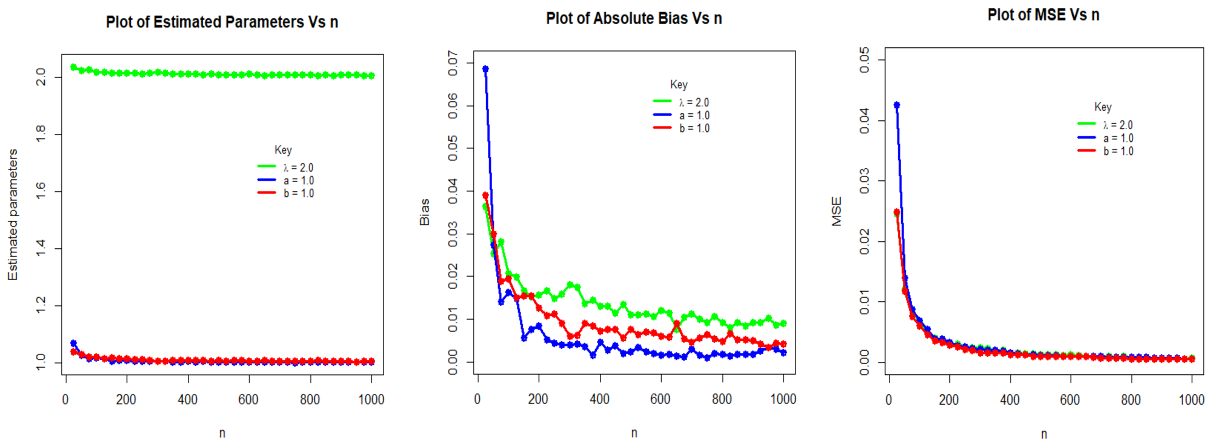

Figure 3: Plots of MLEs, Absolute Biases and MSEs of the NFrExD for
$\lambda = 2.0,\ a = 1.0,\ and\ b = 1.0$

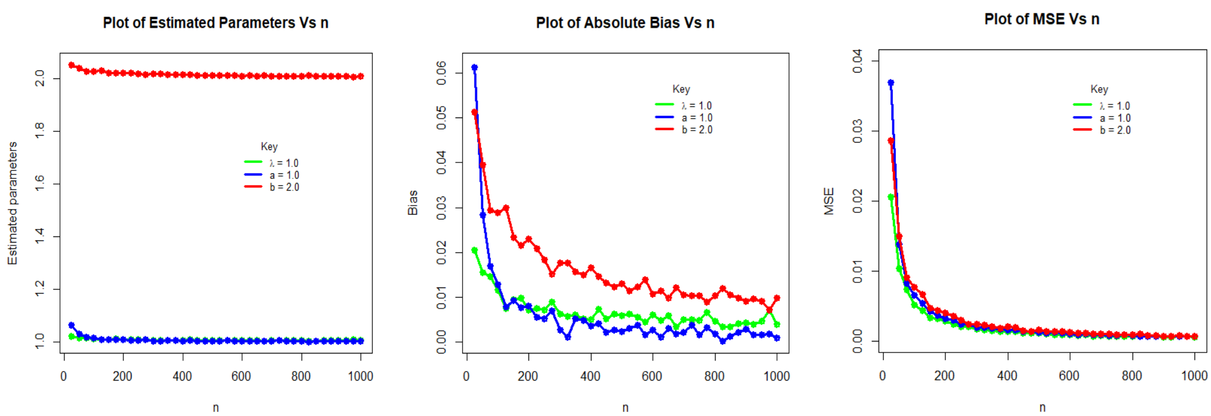

Figure 4: Plots of MLEs, Absolute Biases and MSEs of the NFrExD for
$\lambda = 1.0,\ a = 1.0,\ and\ b = 2.0$

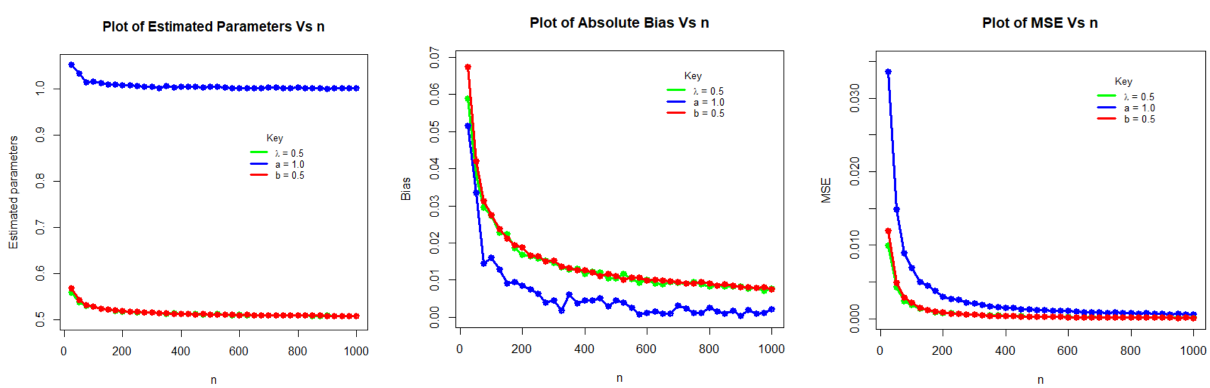

Figure 5: Plots of MLEs, Absolute Biases and MSEs of the NFrExD for
$\lambda = 0.5,\ a = 1.0,\ and\ b = 0.5$

The averages of the MLEs (Mean), their biases (Absolute Bias) and mean square errors (MSEs) for the parameters of the NFrExD are presented in Tables 1–4 and shown in Figures 2–5. Based on the values from the tables and the figures, the study revealed that the average estimates tend to be closer to the true parameters when sample size increases and the biases and mean square errors all decrease as sample size increases which is in agreement with the asymptotic theory of estimation. This is in line with the results of other related studies such as Reis *et al*. (2022), Ahmad *et al*. (2021), Ieren and Abdullahi (2020), Ahmad *et al*. (2020), Alghamdi and Abd El-Raof (2023) as well as Rao and Mbwambo (2019).

8. Applications to Real Life Datasets

In this section, the performance of the NFrExD (NFr-G family) was illustrated using real data sets. For the purpose of comparison, the following models were fitted to the data together with the proposed NFrExD (NFr-G family): Odd Frechet-exponential distribution (OFrExD) from the Odd Frechet-G family of distribution by Haq and Elgarhy (2018), Extended Odd Frechet-exponential distribution (EOFrExD) from the extended odd Frechet-G family of distribution by Nasiru (2018), Generalized Odd Frechet-exponential distribution (GOFrExD) from the generalized odd Frechet-G family of distribution by Marganpoor *et al*. (2020), Exponentiated Frechet-exponential distribution (ExpFrExD) from the exponentiated Frechet-G family of distribution by Baharith and Alamoudi (2021), Lomax-exponential distribution (LomExD) from the Lomax-G family of distribution by Cordeiro *et al.* (2014), Weibull-exponential distribution (WeiExD) from the Weibull-G family of distribution by Bouguignon *et al.* (2014), Lindley-exponential distribution (LinExD) from the Lindley-G family of distribution by Cakmakyapan and Ozel (2016), Transmuted exponential distribution (TrExD) from the Quadratic rank transmutation map by Shaw and Buckley (2007) and Conventional exponential distribution (ExD).

The model selection process was done based on the value of the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan Quin Information Criterion (HQIC) and Kolmogorov-smirnov (K-S) statistics. More information about these measures has also been discussed in Chen and Balakrishnan (1995). Hence, the smaller the value of these measures for a distribution, the better the fit of the distribution.

Data set I: This dataset reflects the body fat percentage of 202 Australian athletes, it was extracted from Oguntunde *et al.,* (2018) and has also been used by Al-Noor and Hadi (2021). The descriptive statistics of this data set are shown in Table 5.

Table 5: Summary Statistics for data set I

parameters	n	Mini	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	202	5.630	8.545	11.650	18.080	13.507	35.520	38.31395	0.75955	-0.1733

Table 5 suggests that the dataset is positively skewed because the coefficient of skewness is more than 0.5.

Table 6: Maximum Likelihood Parameter Estimates for dataset I

Distribution	Parameter Estimates			
NFrExD	$\hat{\lambda}=0.66569702$	$\hat{a}=2.2400303$	$\hat{b}=6.5283361$	
OFrExD	$\hat{\lambda}=0.30195964$	$\hat{b}=0.5160480$		
EOFrExD	$\hat{\lambda}=0.11545505$	$\hat{b}=2.7466403$	$\hat{\alpha}=1.5295591$	
GOFrExD	$\hat{\lambda}=0.32089328$	$\hat{a}=8.8682509$	$\hat{b}=0.5032821$	
ExPFrExD	$\hat{\lambda}=0.26728450$	$\hat{\alpha}=1.0570255$	$\hat{\theta}=2.6073970$	$\hat{\gamma}=2.1947$
LomExD	$\hat{\lambda}=0.05090581$	$\hat{\alpha}=8.7378644$	$\hat{\beta}=6.3253257$	
WeiExD	$\hat{\lambda}=0.02223500$	$\hat{\alpha}=0.3629870$	$\hat{\beta}=3.5710713$	
TrExD	$\hat{\lambda}=0.05983939$	$\hat{\theta}=-0.1875335$		
LinExD	$\hat{\lambda}=0.37581293$	$\hat{\theta}=2.8187918$		
ExD	$\hat{\lambda}=0.07431838$			

Table 7: The statistics ℓ, AIC, CAIC, BIC, HQIC, K-S statistic and P-values for dataset I

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	K-S	P-Value (K-S)
NFrExD	**635.6719**	**1277.344**	**1277.465**	**1287.269**	**1281.359**	**0.085155**	**0.1068**
OFrExD	821.8358	1647.672	1647.732	1654.288	1650.349	0.63505	2.2e-16
EOFrExD	724.2530	1454.506	1454.627	1464.431	1458.522	0.32884	2.2e-16
GOFrExD	646.8376	1299.675	1299.796	1309.600	1303.691	0.18153	3.304e-06
ExPFrExD	654.4528	1316.906	1317.109	1330.139	1322.260	0.20627	6.853e-08
LomExD	737.2964	1480.593	1480.714	1490.518	1484.608	0.32381	2.2e-16
WeiExD	1813.2660	3632.532	3632.653	3642.457	3636.548	0.8469	2.2e-16
TrExD	730.4904	1464.981	1465.041	1471.597	1467.658	0.24943	2.429e-11
LinExD	691.8825	1387.765	1387.825	1394.381	1390.442	0.27435	1.245e-13
ExD	727.8559	1457.712	1457.732	1461.020	1459.050	0.34522	2.2e-16

The following figure presents a plot of estimated PDFs (densities) and CDFs of the fitted models to dataset I.

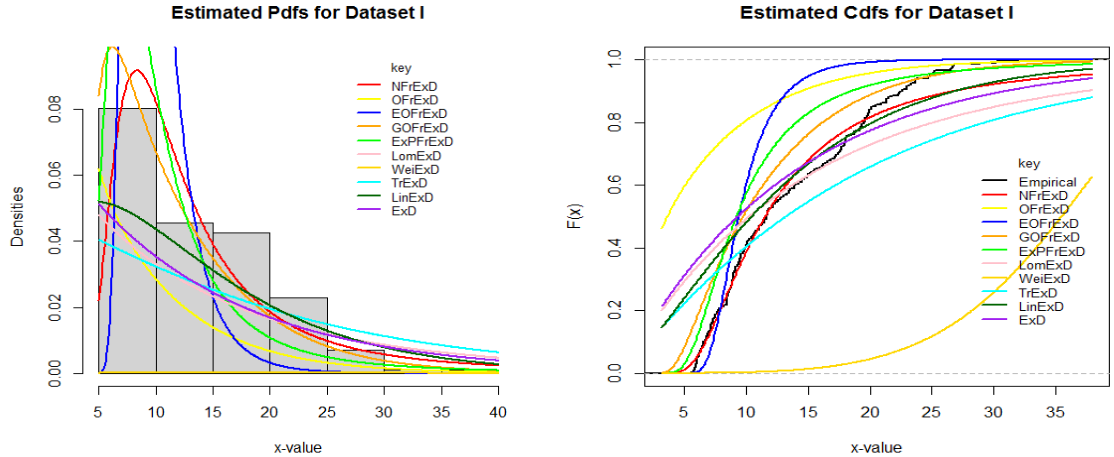

Figure 6: Plots of the estimated densities and CDFs of the fitted distributions to dataset I.

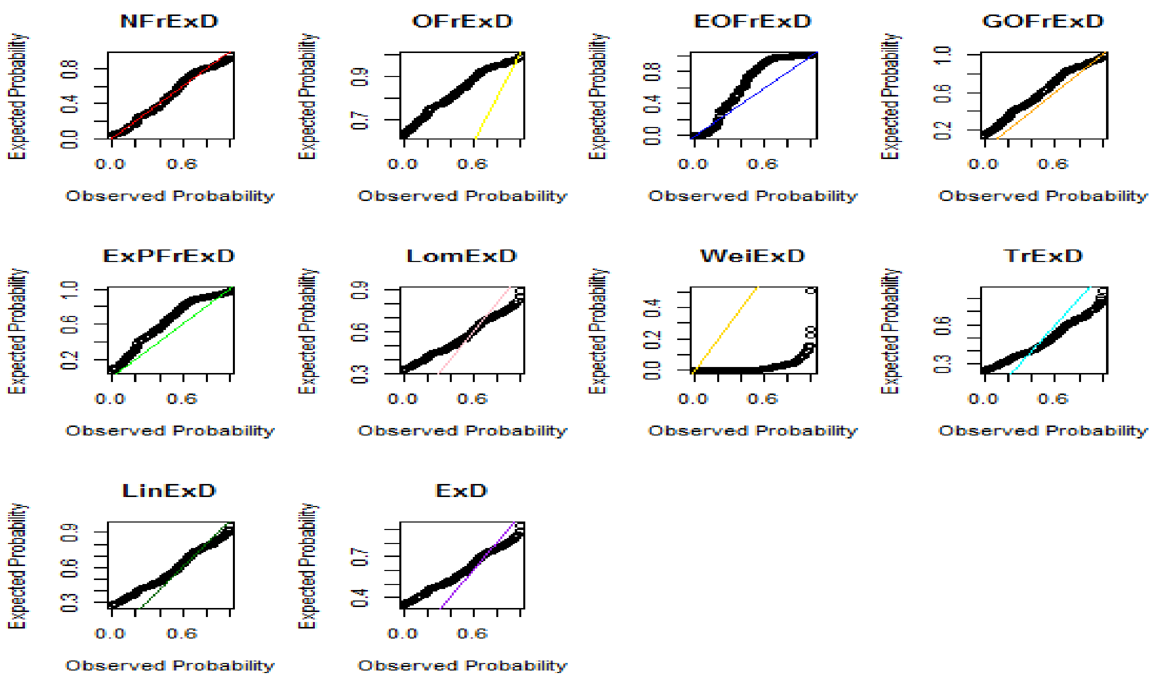

Figure 7: Probability plots for the fitted distributions based on dataset I.

Data Set II. This is a real life dataset and it represents the active repair times (hr) for an airborne communication transceiver. It was reported by Chikara and Folks (1977) and has been used by Dimitrakopoulou *et al.* (2007) and Oluyede *et al.*, (2022). The descriptive statistics of this data set are shown Table 8.

Table 8: Summary Statistics for data set II

n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
46	0.200	0.800	1.750	4.375	3.607	24.500	24.44507	2.88834	8.80256

Table 8 also revealed that the dataset is positively skewed with 2.88834 coefficient of skewness.

Table 9: Maximum Likelihood Parameter Estimates for dataset II

Distribution	Parameter Estimates			
NFrExD	$\hat{\lambda}=4.2128113$	$\hat{a}=1.0299381$	$\hat{b}=4.7474874$	
OfrExD	$\hat{\lambda}=0.3729384$	$\hat{b}=0.5249994$		
EOFrExD	$\hat{\lambda}=0.9624714$	$\hat{b}=1.5395943$	$\hat{\alpha}=6.3964525$	
GOFrExD	$\hat{\lambda}=0.5022227$	$\hat{a}=1.0017863$	$\hat{b}=0.6066838$	
ExPFrExD	$\hat{\lambda}=0.7536618$	$\hat{\alpha}=2.1499805$	$\hat{\theta}=0.7021404$	$\hat{\gamma}=1.871423$
LomExD	$\hat{\lambda}=0.8616947$	$\hat{\alpha}=3.5685917$	$\hat{\beta}=8.0645112$	
WeiExD	$\hat{\lambda}=0.1618339$	$\hat{\alpha}=4.7212988$	$\hat{\beta}=0.4238973$	
TrExD	$\hat{\lambda}=0.2004948$	$\hat{\theta}=0.7397945$		
LinExD	$\hat{\lambda}=3.2728306$	$\hat{\theta}=9.6556276$		
ExD	$\hat{\lambda}=0.2772698$			

Table 10: The statistics ℓ, AIC, CAIC, BIC, HQIC, K-S statistic and P-values for dataset II

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	K-S	P-Value (K-S)
NFrExD	**100.70286**	**207.4057**	**207.9771**	**212.8916**	**209.4608**	**0.085649**	**0.8885**
OFrExD	102.94834	209.8967	210.1758	213.5540	211.2667	0.18756	0.07861
EOFrExD	102.84175	211.6835	212.2549	217.1694	213.7386	0.12175	0.5029
GOFrExD	102.04478	210.0896	210.6610	215.5755	212.1446	0.10658	0.673
ExPFrExD	99.86414	207.7283	208.7039	215.0429	210.4684	0.077359	0.9459
LomExD	102.95498	211.9100	212.4814	217.3959	213.9650	0.12603	0.4581
WeiExD	278.13531	562.2706	562.8420	567.7565	564.3257	0.76534	2.2e-16
TrExD	103.67541	211.3508	211.6299	215.0081	212.7209	0.11574	0.5687
LinExD	105.74635	215.4927	215.7718	219.1500	216.8627	0.173	0.1274
ExD	105.00621	212.0124	212.1033	213.8411	212.6974	0.15974	0.191

The following figure presents a histogram and estimated densities and CDFs of the fitted models to dataset II.

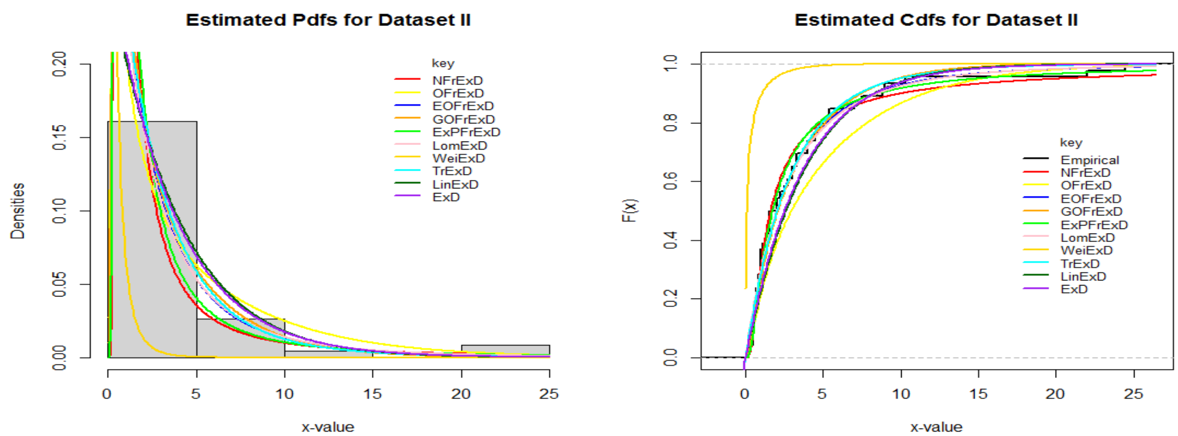

Figure 8: Plots of the estimated densities and CDFs of the fitted distributions to dataset II.

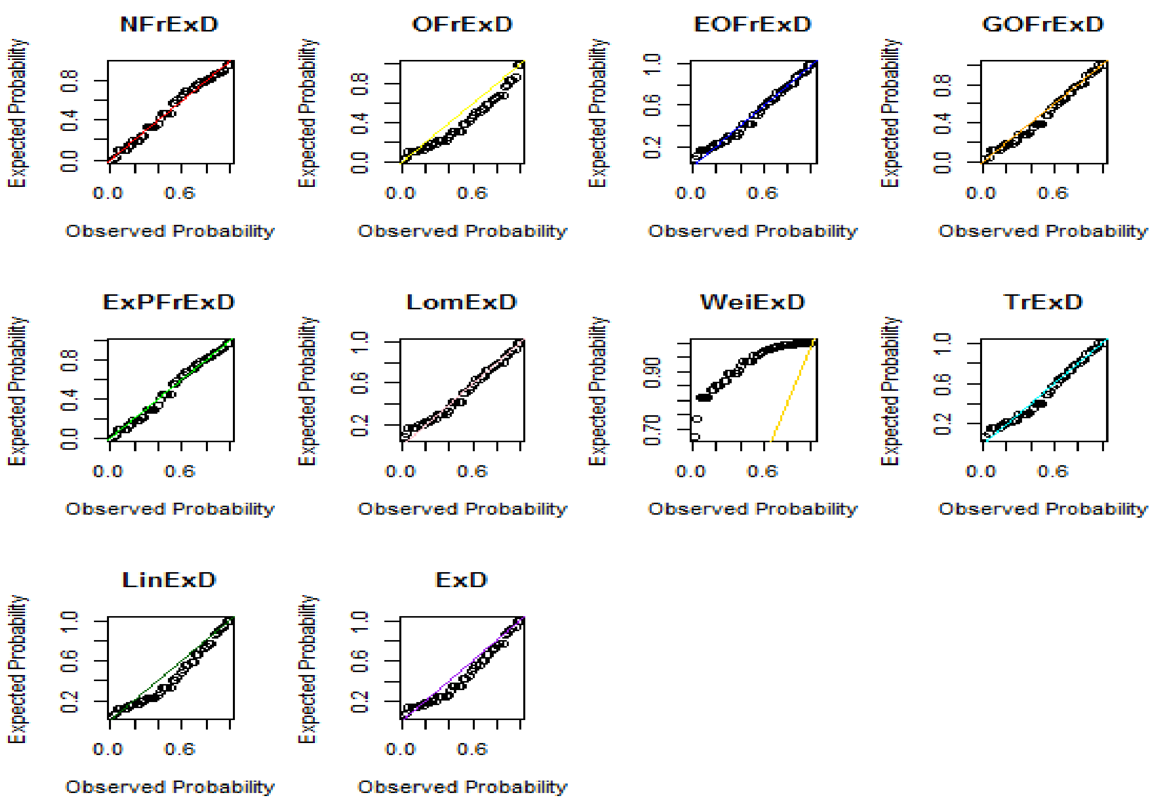

Figure 9: Probability plots for the fitted distributions based on dataset II.

Data Set III. The third data set is on shape measurements of 48 rock samples from a petroleum reservoir. This data was extracted from BP research, image analysis by Ronit Katz, U Oxford and has been used for analysis by Javanshiri *et al*. (2015) and Alhaji and Haruna (2023). Its summary is presented in Table 11.

Table 11: Summary Statistics for data set III

parameters	n	Mini	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	48	0.09033	0.16226	0.19880	0.26267	0.21811	0.46413	0.00697	1.16939	1.10986

The information in Table 11 shows that the dataset is positively skewed with 1.16939 coefficient of skewness.

Table 12: Maximum Likelihood Parameter Estimates for dataset III

Distribution	Parameter Estimates			
NFrExD	$\hat{\lambda}=7.55375719$	$\hat{a}=3.08200905$	$\hat{b}=1.316282$	
OFrExD	$\hat{\lambda}=3.95484073$	$\hat{b}=2.17736507$		
EOFrExD	$\hat{\lambda}=0.03395197$	$\hat{b}=0.04135533$	$\hat{\alpha}=4.027266$	
GOFrExD	$\hat{\lambda}=6.24285622$	$\hat{a}=1.70475473$	$\hat{b}=1.590581$	
ExPFrExD	$\hat{\lambda}=5.24386237$	$\hat{\alpha}=6.06459001$	$\hat{\theta}=1.357340$	$\hat{\gamma}=1.931753$
LomExD	$\hat{\lambda}=3.93788555$	$\hat{\alpha}=9.36332482$	$\hat{\beta}=7.911336$	
WeiExD	$\hat{\lambda}=0.41926301$	$\hat{\alpha}=8.37402316$	$\hat{\beta}=1.245374$	
TrExD	$\hat{\lambda}=6.68054539$	$\hat{\theta}=-0.99617300$		
LinExD	$\hat{\lambda}=0.44191564$	$\hat{\theta}=0.06740521$		
ExD	$\hat{\lambda}=4.58488876$			

Table 13: The statistics ℓ, AIC, CAIC, BIC, HQIC, K-S statistic and P-values for dataset III

Distribution	$\hat{\ell}$	*AIC*	*CAIC*	*BIC*	*HQIC*	**K-S**	**P-Value (K-S)**
NFrExD	**-56.346362**	**-106.6927**	**-106.1473**	**-101.0791**	**-104.5713**	**0.067173**	**0.9819**
OFrExD	-54.902172	-105.8043	-105.5377	-102.0619	-104.3901	0.069344	0.9751
EOFrExD	-5.071422	-4.142844	-3.597389	1.470759	-2.021455	0.8645	2.2e-16
GOFrExD	-54.912199	-103.8244	-103.2789	-98.21079	-101.703	0.071479	0.9669
ExPFrExD	-55.155464	-102.3109	-101.3807	-94.82612	-99.48241	0.10424	0.6741
LomExD	-22.943690	-39.887381	-39.341926	-34.273778	-37.765992	0.38234	1.608e-06
WeiExD	-4.771455	-3.542911	-2.997456	2.070692	-1.421522	0.3807	1.813e-06
TrExD	-39.144179	-74.288357	-74.021691	-70.545955	-72.874098	0.28105	0.001018
LinExD	-32.350675	-60.701349	-60.434682	-56.958947	-59.287090	0.25971	0.003083
ExD	-25.092746	-48.185492	-48.098535	-46.314291	-47.478362	0.38583	1.243e-06

The following figure presents a histogram and estimated densities and CDFs of the fitted models to dataset III.

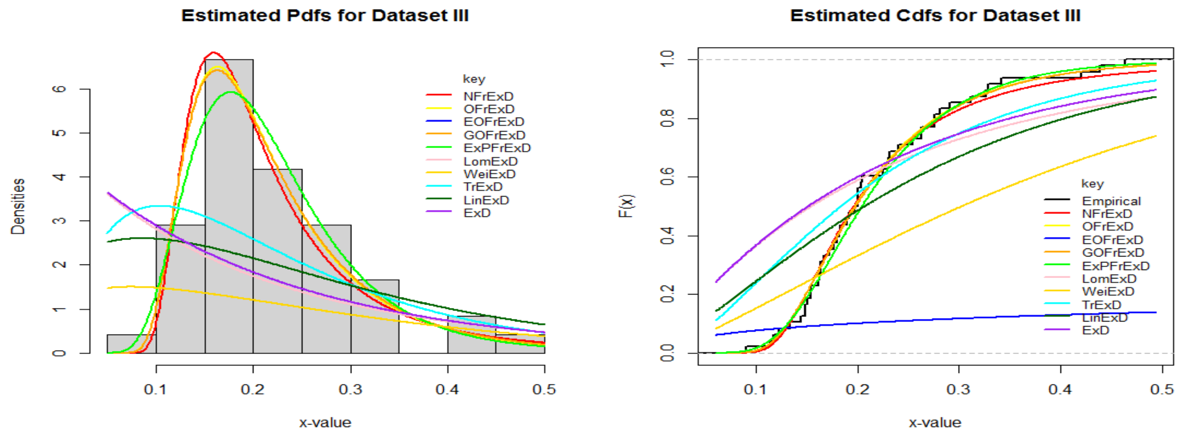

Figure 10: Plots of the estimated densities and CDFs of the fitted distributions to dataset III.

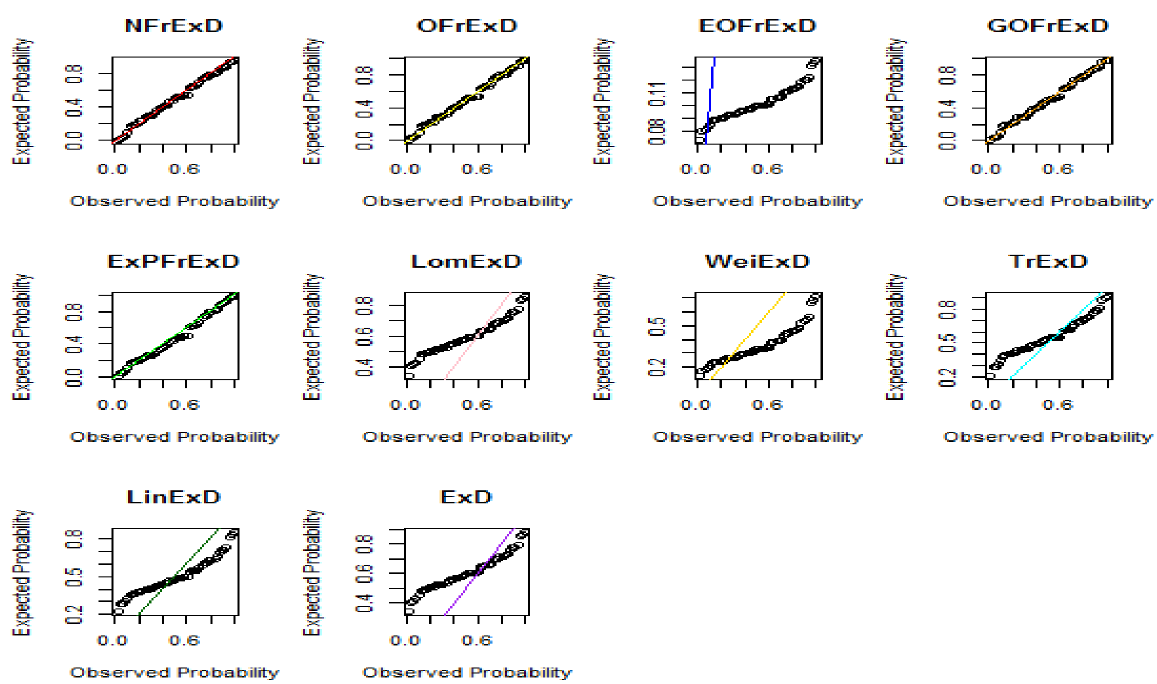

Figure 11: Probability plots for the **TEN** fitted distributions based on dataset III.

Tables 6, 9 and 12 present the maximum likelihood parameter estimates of the ten fitted distributions based on dataset I, dataset II and dataset III respectively and Tables 7, 10 and 13 list the values of AIC, CAIC, BIC, HQIC and K-S with p-value for the fitted distributions based on dataset I, dataset II and dataset III respectively. Similarly, the plots of the fitted densities and cumulative distribution functions of the NFrExD with those of competing distributions for dataset I, dataset II and dataset III are displayed in Figures 6, 8 and 10 respectively and the probability plots of the fitted distributions for dataset I, dataset II and dataset III are displayed in figures 7, 9 and 11 respectively. The values of AIC, CAIC, BIC, HQIC and K-S in Table 7 which is for dataset I are smaller for the proposed NFrExD compared to the other nine fitted probability distributions. This indicates that the NFrExD from the proposed New Frechet-G family performs better than the other fitted distributions. This outstanding performance of the NFrExD is followed by GOFrExD and ExpFrExD coming second and third respectively. This result for dataset I which based on the values of AIC, CAIC, BIC, HQIC and K-S is in agreement with the plots of the estimated PDFs and CDFs of the fitted distributions to dataset I displayed in Figure 6 as well as the probability plots Figure 7. Also, considering the results for dataset II based on the values of AIC, CAIC, BIC, HQIC and K-S from Table 10, it is revealed that the NFrExD has lower values of the AIC, CAIC, BIC, HQIC and K-S as compared to the other nine fitted distributions. This means that the proposed distribution (NFrExD) fits the data (dataset II) better than the other distributions from the other families of distribution. The result for dataset II which based on AIC, CAIC, BIC, HQIC and K-S was confirmed by the plots of the estimated PDFs and CDFs of the fitted distributions to dataset II displayed in Figure 8 as well as the probability plots Figure 9. More so, the proposed NFrExD has minimum values of AIC, CAIC, BIC, HQIC and K-S from Table 13 for dataset III which shows that it has the best fitting performance compared to the other fitted distributions. This means that the proposed distribution (NFrExD) fits dataset III better than the other nine distributions. The result for dataset III from Table 13 which based on AIC, CAIC, BIC, HQIC and K-S was confirmed by the plots of the estimated PDFs and CDFs of the fitted distributions to dataset III displayed in Figure 10 as well as the probability plots in Figure 11. This result shows that the NFr-G family is a good family for extending other probability distributions and also confirms the fact that compound or extended probability distributions are better than the standard ones or the baseline distributions as previously reported by Bhat *et al*. (2023), Bouguignon *et al.* (2014), Ieren and Balogun (2021), Umar *et al*. (2021), Haq and Elgarhy (2018), Cordeiro et al. (2019), Anzagra *et al*. (2022), Benchiha *et al*. (2023), Gabanakgosi and Oluyede (2023), Alotaibi *et al*. (2022), Fayomi *et al*. (2023), Mohammad (2024), and so on.

9. Conclusion

This article has developed a new family of distributions known as "New Frechet-G family" with special models and structural properties as well as the maximum likelihood estimation of the unknown parameters. The shapes of the densities of the special models for some chosen parameter values were also discussed. The plots of the CDFs of the special models were also presented and discussed. Monte Carlo simulation study was conducted to check the behaviour of the parameters. The plots of the PDFs of the NFrExD and NFrWeiD show that the proposed family of distribution is skewed and flexible and that its shape depends on the values of the parameters. Also, the shapes can be classified as decreasing, unimodal, right-skewed and asymmetrical shapes. Also, the plots of the CDF confirmed that the special models and the entire NFr-G family is a valid family of probability distribution. From the results of the simulation study of NFr-G family through NFrExD, it is revealed that the average estimates tend to be closer to the true parameters when sample size increases and the biases and mean square errors all decrease as sample size increases which is in agreement with the statistical principle of estimation. It is also discovered that the values of the parameters have no effect on the estimators of any other parameter because changing the values of the parameters alone does not change the direction of the result or the bias and mean squared errors. Applications of the proposed distribution using three real life datasets revealed that the NFrExD fits the three datasets more appropriate and has better fitting performance compared to the other competing models considered in this study.

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