# EXPONENTIATED GENERALIZED BURR XII DISTRIBUTION: PROPERTIES AND APPLICATIONS

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**Abstract**

This study introduces a new four-parameter distribution, the Exponentiated Generalized Burr XII (EGBXII) distribution. The model was developed by combining the classical Burr XII distribution with the Exponentiated Generalized family, offering enhanced flexibility. Its probability density function (pdf) exhibits desirable features, including unimodal and inverted bathtub shapes. The hazard rate function can represent both increasing and decreasing patterns, making it versatile for modeling diverse real-world phenomena. Key properties of the distribution, such as moments, the moment-generating function, skewness, and kurtosis, are derived. The parameters of the distribution were estimated using the maximum likelihood method. A simulation study was conducted to evaluate the behavior of these estimated parameters. Finally, the proposed model was applied to two real-world datasets, where it demonstrated superior performance in terms of efficiency and consistency compared to other existing models, as evaluated using comparative criteria, including Akaike information Criterion (AIC), Bayesian information criterion (BIC), Hannan Quinn information criterion (HQIC), Corrected Akaike Information Criterion (CAIC), and the Kolmogorov-Smirnov (KS) test.

**Keywords:** Exponentiated Generalized G., Burr XII, Moments, Maximum likelihood estimation, AIC.

## 1. Introduction

Probability distributions are fundamental tools in statistics and data analysis, enabling the modeling and understanding of random processes. They describe the likelihood of different outcomes and are widely applied across various fields, including engineering, finance, healthcare, and environmental science. Classical distributions, such as the Weibull, exponential,



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and gamma distributions, are commonly used for modeling data, predicting outcomes, and conducting analyses (Qayoom *et al*. 2024).

However, while these classical distributions are versatile, they often struggle to fit complex or highly skewed datasets effectively. Their inherent structures may not capture critical data characteristics, such as heavy tails or multimodality, which can lead to suboptimal model performance and inaccurate conclusions. Consequently, there is a need for more flexible probability distributions that can address these challenges and improve analytical accuracy.

To meet this need, researchers have developed new and more flexible distributions by modifying existing ones through distribution families. These families introduce additional parameters, such as scale, shape, or location, thereby enhancing the flexibility and applicability of classical distributions. Examples of such families in the literature include the shifted Gompertz-G family of distributions by Eghwerido and Agu (2021), the transmuted Burr X-G Family by Al-Babtain *et al.* (2021), the Odd Gompertz-G family by Kajuru *et al.* (2023), the new Generalized Odd Frechet-G family by Sadiq *et al.* (2023), the Sine Type II Topp-Leone G family by Isa *et al.* (2023), the alpha-sine-G family by Benchiha *et al.* (2023), the new Fréchet-G family by Ieren *et al.* (2024), the tangent exponential-G family by Hussam *et al.* (2024), and the Generalized Gompertz-G family by Garba *et al.* (2024).

The classical Burr XII distribution, introduced by Burr in 1942, is widely utilized in fields such as reliability studies, actuarial science, and various other domains. Several extensions have been proposed to improve its flexibility, including the Kumaraswamy Burr XII distribution by Paranaiba *et al.* (2013), the odd exponentiated half-logistic Burr XII distribution by Aldahlan and Afify (2018), the modified Burr XII distribution by Jamal *et al.* (2020), the exponentiated exponential Burr XII distribution by Badr and Ijaz (2021), the harmonic mixture Burr XII distribution by Cloo *et al.* (2023), the odd logistic Burr XII distribution by Sandos and Pescim (2023), the Marshall–Olkin Weibull–Burr XII distribution by Alsadat *et al.* (2023), the bell Burr XII distribution by Alanzi et al. (2023), the type II Topp-Leone Burr XII distribution by Ogunde and Adeniji (2023), the new weighted Burr XII by Nafo *et al.* (2024), the sine type II Topp-Leone Burr XII distribution by Isa *et al.* (2024), and the toppleone exponentiated Burr XII distribution by Isa *et al.* (2024) .



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In this work, we introduce a new extension of the Burr XII distribution by compounding it with the Exponentiated Generalized family of distributions proposed by Cordeiro *et al.* (2013), resulting in a more versatile model. This new distribution, called the Exponentiated Generalized Burr XII (EGBXII) distribution, enhances the flexibility of the Burr XII distribution, making it more effective at fitting complex datasets across various domains. The motivation behind this extension is to address the need for more sophisticated probability distributions capable of handling intricate real-world data. Through rigorous theoretical analysis and empirical validation, this study demonstrates the efficacy of the EGBXII distribution in addressing these modeling challenges, offering more accurate and reliable statistical inference across a wide range of applications.

## 2. Methods and Methodology

### 2.1 Exponentiated Generalized Family of Distributions

The cumulative distribution function (CDF) of the Exponentiated Generalized G family of distribution is given by:

$$F(x) = \left\{1 - [1 - G(x)]^{\lambda}\right\}^{\theta} \tag{1}$$

and the corresponding probability density function (PDF) is given by:

$$f(x) = \lambda\theta g(x)[1 - G(x)]^{\lambda-1}\left\{1 - [1 - G(x)]^{\lambda}\right\}^{\theta-1};\ for\ x\ > 0 \tag{2}$$

where $\lambda\ >\ 0\ and\ \theta\ >\ 0$ are the shape parameters.

### 2.1 Burr XII Distribution

The CDF and the PDF of the proposed Exponentiated Generalized Burr XII distribution is given in equation (3) and (4) respectively as follows:

$$G(x) = 1 - \frac{1}{(1 + x^{\alpha})^{\beta}} \tag{3}$$

and

$$g(x) = \frac{\alpha\beta x^{\alpha-1}}{(1 + x^{\alpha})^{\beta+1}}\ ;\ for\ x\ >\ 0 \tag{4}$$



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where $\alpha > 0$ $and$ $\beta > 0$ are the shape parameters respectively.

**2.3 The Proposed Exponentiated Generalized Burr XII (EGBXII) Distribution**

The cumulative distribution function (CDF) of the proposed EGBXII distribution is derived by substituting the CDF of the Burr XII distribution from Equation (3) into Equation (1), yielding:

$$F(x) = \left\{1 - \left[1 - \left(1 - \frac{1}{(1 + x^{\alpha})^{\beta}}\right)\right]^{\lambda}\right\}^{\theta} \tag{5}$$

By substituting equations (3) and (4) into equation (2), the PDF of the proposed EGBXII is given in equation (6) as follows:

$$f(x) = \frac{\alpha\beta\lambda\theta x^{\alpha-1}}{(1 + x^{\alpha})^{\beta+1}}\left[1 - \left(1 - \frac{1}{(1 + x^{\alpha})^{\beta}}\right)\right]^{\lambda-1}\left\{1 - \left[1 - \left(1 - \frac{1}{(1 + x^{\alpha})^{\beta}}\right)\right]^{\lambda}\right\}^{\theta-1} \tag{6}$$

The survival function, hazard function, and quantile function are presented in equations (7) to (9), respectively. These functions are derived by substituting the expressions from equations (5) and (6) into their respective formulas.

$$S(x) = 1 - F(x) = 1 - \left\{1 - \left[1 - \left(1 - \frac{1}{(1 + x^{\alpha})^{\beta}}\right)\right]^{\lambda}\right\}^{\theta} \tag{7}$$

$$h(x) = \frac{f(x)}{1 - F(x)}$$

$$= \frac{\frac{\alpha\beta\lambda\theta x^{\alpha-1}}{(1 + x^{\alpha})^{\beta+1}}\left[1 - \left(1 - \frac{1}{(1 + x^{\alpha})^{\beta}}\right)\right]^{\lambda-1}\left\{1 - \left[1 - \left(1 - \frac{1}{(1 + x^{\alpha})^{\beta}}\right)\right]^{\lambda}\right\}^{\theta-1}}{1 - \left\{1 - \left[1 - \left(1 - \frac{1}{(1 + x^{\alpha})^{\beta}}\right)\right]^{\lambda}\right\}^{\theta}} \tag{8}$$

$$Q(u) = F^{-1}(u) = \left\{1 - \left[1 - \left(1 - \left(1 - u^{\frac{1}{\theta}}\right)^{\frac{1}{\lambda}}\right)\right]^{-\frac{1}{\beta}} - 1\right\}^{\frac{1}{\alpha}} \tag{9}$$

Figures 1 to 4 illustrate the characteristics of the proposed EGBXII distribution. The probability density function (PDF) is right-skewed and unimodal. The CDF ranges from 0 to 1, as expected for any valid probability distribution. The survival function declines from 1 to 0, reflecting the



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decreasing likelihood of survival over time. Lastly, the hazard function shows an increasing and then decreasing failure rate pattern, suggesting that the risk of failure varies over time.

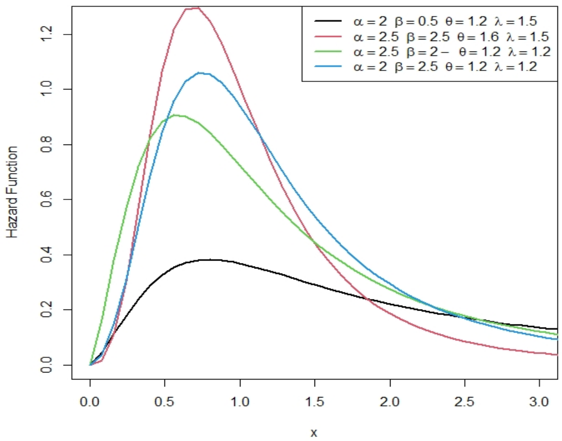


Figure 1: PDF plot of EGBXII distribution

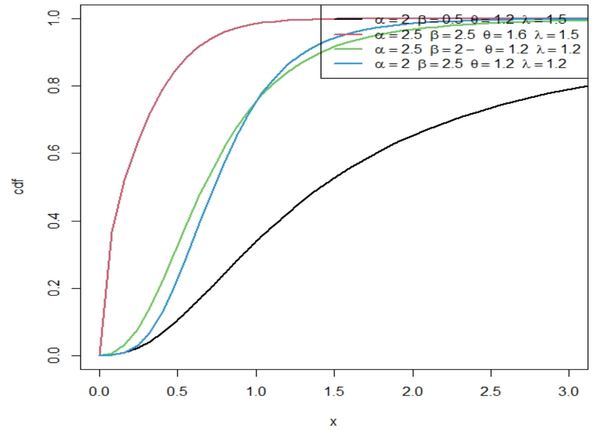


Figure 2: CDF plot of EGBXII distribution

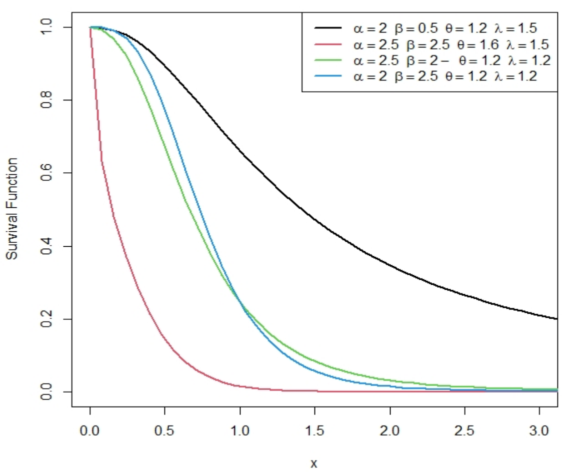


Figure 3: Survival plot of EGBXII distribution

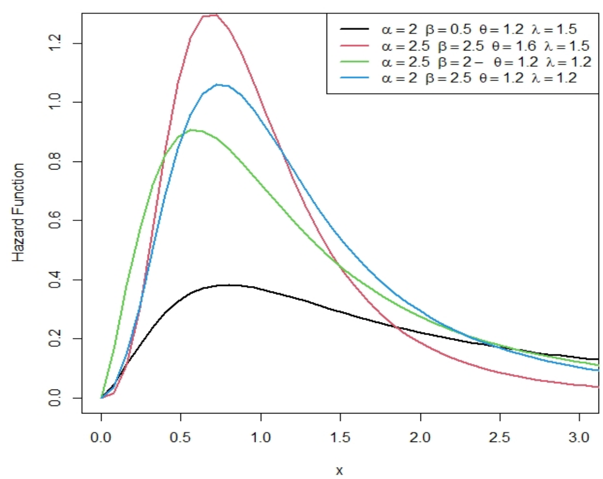


Figure 4: HRF plot of EGBXII distribution

## 3. Useful Expansion

The CDF of the EGBXII distribution can be expanded using power series expansion as follows:

$$\left\{1-\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda}\right\}^{\theta}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\theta}{i}\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda i}$$



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$$\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda i}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\lambda i}{j}\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{j}$$

$$\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{j}=\sum_{k=0}^{\infty}(-1)^{k}\binom{j}{k}(1+x^{\alpha})^{-\beta k}$$

$$F(x)=\sum_{i=0}^{\infty}(-1)^{i+j+k}\binom{\theta}{i}\binom{\lambda i}{j}\binom{j}{k}(1+x^{\alpha})^{-\beta k}$$

$$F(x)=\sum_{i,j,k=0}^{\infty}\Psi_{i,j,k}(1+x^{\alpha})^{-\beta k} \tag{10}$$

where

$$\Psi_{i,j,k}=(-1)^{i+j+k}\binom{\theta}{i}\binom{\lambda i}{j}\binom{j}{k}$$

The PDF can also be expanded as follows:

$$\left\{1-\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda}\right\}^{\theta-1}=\sum_{m=0}^{\infty}(-1)^{m}\binom{\theta-1}{m}\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda m}$$

$$\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda m+\lambda-1}=\sum_{n=0}^{\infty}(-1)^{n}\binom{\lambda m+\lambda-1}{n}\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{n}$$

$$\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)^{n}=\sum_{p=0}^{\infty}(-1)^{p}\binom{n}{p}(1+x^{\alpha})^{-\beta p}$$

$$f(x)=\sum_{m,n,p=0}^{\infty}(-1)^{m+n+p}\theta\lambda\alpha\beta\binom{\theta-1}{m}\binom{\lambda m+\lambda-1}{n}\binom{n}{p}x^{\alpha-1}(1+x^{\alpha})^{-(\beta p+\beta+1)}$$

Therefore, the PDF can be expressed as follows:

$$f(x)=\sum_{m,n,p=0}^{\infty}\Psi_{m,n,p}\theta\lambda\alpha\beta\, x^{\alpha-1}(1+x^{\alpha})^{-(\beta p+\beta+1)} \tag{11}$$



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where $\Psi_{m,n,p} = (-1)^{m+n+p} \binom{\theta-1}{m} \binom{\lambda m + \lambda - 1}{n} \binom{n}{p}$

## 4. Statistical Properties

Some of the statistical properties of the proposed EGBXII distribution such as the moment generating function, $r^{th}$ moment, measures of skewness and kurtosis, and probability weighted moment were derived in this section.

### 4.1 Moment Generating Function

The moment generating function of a random variable X is obtained using the formula:

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

The moment generating function of the proposed EGBXII distribution is given by:

$$M_x(t) = \sum_{m,n,p=0}^{\infty} \Psi_{m,n,p} \theta\lambda\alpha\beta \int_0^{\infty} e^{tx}\, x^{r+\alpha-1}\, (1 + x^{\alpha})^{-(\beta p+\beta+1)} dx$$

Integrating and simplifying, the integral becomes:

$$M_x(t) = \sum_{m,n,p=0}^{\infty} \Psi_{m,n,p} \theta\lambda\beta \frac{t^q}{q!} B(q + 1, \beta p + \beta - q) \tag{12}$$

### 4.2 Moment

The $r^{th}$ moment of the EGBXII distribution is obtained as follows:

$$\mu_r = \sum_{m,n,p=0}^{\infty} \Psi_{m,n,p} \theta\lambda\alpha\beta \int_0^{\infty} x^{r+\alpha-1}\, (1 + x^{\alpha})^{-(\beta p+\beta+1)} dx$$

$$\mu_r = \sum_{m,n,p=0}^{\infty} \Psi_{m,n,p} \theta\lambda\beta\;\; B\left(\frac{r}{\alpha}, \beta p + 1 - \frac{r}{\alpha}\right) \tag{13}$$

The first, second, third and the forth moments are obtained by substituting $r = 1, 2, 3,$ and 4 into equation (13) as follows:

$$\mu_1 = \sum_{m,n,p=0}^{\infty} \Psi_{m,n,p}\theta\lambda\beta \ B\left(\frac{1}{\alpha}, \beta p + 1 - \frac{1}{\alpha}\right) \tag{14}$$

$$\mu_2 = \sum_{m,n,p=0}^{\infty} \Psi_{m,n,p}\theta\lambda\beta \ B\left(\frac{2}{\alpha}, \beta p + 1 - \frac{2}{\alpha}\right) \tag{15}$$

$$\mu_3 = \sum_{m,n,p=0}^{\infty} \Psi_{m,n,p}\theta\lambda\beta \ B\left(\frac{3}{\alpha}, \beta p + 1 - \frac{3}{\alpha}\right) \tag{16}$$

$$\mu_4 = \sum_{m,n,p=0}^{\infty} \Psi_{m,n,p}\theta\lambda\beta \ B\left(\frac{4}{\alpha}, \beta p + 1 - \frac{4}{\alpha}\right) \tag{17}$$

**4.3 Measure of Skewness and Kurtosis**

Skewness and kurtosis of the EGBXII distribution denoted by $\beta_1$ and $\beta_2$ respectively are expressed in form of $\mu_2, \mu_3$ and $\mu_4$ as follows:

$$\beta_1 = \frac{(\mu_3)^2}{(\mu_2)^3} = \frac{\left(\sum_{m,n,p=0}^{\infty} \Psi_{m,n,p}\theta\lambda\beta \ B\left(\frac{3}{\alpha}, \beta p + 1 - \frac{3}{\alpha}\right)\right)^2}{\left(\sum_{m,n,p=0}^{\infty} \Psi_{m,n,p}\theta\lambda\beta \ B\left(\frac{2}{\alpha}, \beta p + 1 - \frac{2}{\alpha}\right)\right)^3} \tag{18}$$

and

$$\beta_1 = \frac{\mu_4}{(\mu_2)^2} = \frac{\sum_{m,n,p=0}^{\infty} \Psi_{m,n,p}\theta\lambda\beta \ B\left(\frac{4}{\alpha}, \beta p + 1 - \frac{4}{\alpha}\right)}{\left(\sum_{m,n,p=0}^{\infty} \Psi_{m,n,p}\theta\lambda\beta \ B\left(\frac{2}{\alpha}, \beta p + 1 - \frac{2}{\alpha}\right)\right)^2} \tag{19}$$

**4.4 Probability Weighted Moment**

The probability weighted moment of a random variable $x$ is expressed as:

$$M_{r,s,t} = E\left[X^r(F(X)^s\big(1 - F(x)\big)^t\right] \tag{20}$$

For the EGBXII distribution, the Probability Weighted Moment is given by:



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$$M_{r,s,t} = \sum_{m,n,p=0}^{\infty} \Psi_{m,n,p}\theta\lambda\beta \;\; B\left(\frac{1}{\alpha}, \beta p + \beta + 1 - \frac{1}{\alpha}\right) \tag{21}$$

## 5. Parameter Estimation

### 5.1 Maximum Likelihood Estimation

The unknown parameters of the EGBXII distribution were estimated using the method of maximum likelihood. The sample log-likelihood function of the EGBXII model is given as follows:

$$\ell = n\log(\alpha) + n\log(\beta) + n\log(\lambda) + n\log(\theta) + (\alpha+1)\log(x) - (\beta-1)\sum_{i=1}^{n}\log(1+x^{\alpha})$$
$$+(\lambda-1)\sum_{i=1}^{n}\log\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right] + (\theta-1)\sum_{i=1}^{n}\log\left\{1-\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda}\right\} \tag{22}$$

Differentiating equation (22) with respect to $\alpha$ gives:

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} + \log(x) - (\beta+1)\;\sum_{i=1}^{n}\frac{x^{\alpha}\log(x)}{(1+x^{\alpha})} - \beta(\lambda-1)\sum_{i=1}^{n}x^{\alpha}\log(x)(1+x^{\alpha})^{\beta-1}$$
$$\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right] + (\theta-1)\sum_{i=1}^{n}\frac{\lambda\beta x^{\alpha}\log(x)(1+x^{\alpha})^{\beta-1}\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda}}{\left\{1-\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda}\right\}} \tag{23}$$

Differentiating equation (22) with respect to $\beta$ gives:

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log(1+x^{\alpha}) - (\lambda-1)\sum_{i=1}^{n}\frac{(1+x^{\alpha})^{-\beta}\log(1+x^{\alpha})}{\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]}$$
$$+(\theta-1)\sum_{i=1}^{n}\frac{\lambda(1+x^{\alpha})^{-\beta}\log(1+x^{\alpha})\;\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda-1}}{\left\{1-\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda}\right\}} \tag{24}$$

Differentiating equation (22) with respect to $\lambda$ gives:

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right] + (\theta-1)\sum_{i=1}^{n}\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda}$$



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$$\times \log\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right] \tag{25}$$

Differentiating equation (22) with respect to $\beta$ gives:

$$\frac{\partial \ell}{\partial \beta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log\left\{1-\left[1-\left(1-\frac{1}{(1+x^{\alpha})^{\beta}}\right)\right]^{\lambda}\right\} \tag{26}$$

The solution to Equation (23) to (26) gives the maximum likelihood estimates of the parameters $\alpha, \beta, \lambda$ and $\theta$ respectively.

## 6. Monte Carlo Simulation

A Monte Carlo simulation was conducted to evaluate the consistency of the parameter estimates for the proposed EGBXII distribution across different sample sizes. The simulation tested four distinct sets of parameter values. The first set used initial parameter values of $\alpha$ = 1, $\beta$ = 1, $\lambda$ = 1, and $\theta$ = 1. The second set of parameters used $\alpha$ = 0.5, $\beta$ = 0.3, $\lambda$ = 0.8, and $\theta$ = 0.7. The third set used $\alpha$ = 1.2, $\beta$ = 1.2, $\lambda$ = 1.2, and $\theta$ = 1.2, while the fourth set employed $\alpha$ = 0.9, $\beta$ = 0.9, $\lambda$ = 0.9, and $\theta$ = 0.9.

The simulation study followed these steps:

(i) Generate N = 2000 samples with sizes n = 20, 50, 70, 100 and 150

(ii) Compute the maximum likelihood estimates for the model.

(iii) Calculate the root mean square errors (RMSE) and biases.



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**Table 1.** Simulation results with varying values for the four parameters.

| N | Properties | α=1 | β =1 | λ=1 | θ =1 | α=0.5 | β =0.3 | λ=0.8 | θ =0.7 |
|---|---|---|---|---|---|---|---|---|---|
| 20 | Mean | 1.0509 | 1.0272 | 1.1038 | 1.1554 | 0.6010 | 0.3340 | 0.7438 | 0.9316 |
| | Bias | 0.0509 | 0.0272 | 0.1038 | 0.1554 | 0.1010 | 0.0340 | -0.0562 | 0.2316 |
| | RMSE | 0.3407 | 0.2769 | 0.2626 | 0.3829 | 0.3190 | 0.1961 | 0.2291 | 0.4084 |
| 50 | Mean | 1.0134 | 1.0035 | 1.0806 | 1.1008 | 0.5383 | 0.3051 | 0.7649 | 0.8694 |
| | Bias | 0.0134 | 0.0035 | 0.0806 | 0.1008 | 0.0383 | 0.0051 | -0.0351 | 0.1694 |
| | RMSE | 0.2461 | 0.2241 | 0.2216 | 0.3010 | 0.1876 | 0.1457 | 0.1671 | 0.2867 |
| 70 | Mean | 1.0118 | 0.9981 | 1.0674 | 1.0809 | 0.5254 | 0.2943 | 0.7757 | 0.8568 |
| | Bias | 0.0118 | -0.0019 | 0.0674 | 0.0809 | 0.0254 | -0.0057 | -0.0243 | 0.1568 |
| | RMSE | 0.2245 | 0.2156 | 0.1916 | 0.2853 | 0.1545 | 0.1265 | 0.1387 | 0.2525 |
| 100 | Mean | 1.0008 | 0.9971 | 1.0637 | 1.0680 | 0.5145 | 0.2873 | 0.7834 | 0.8494 |
| | Bias | 0.0008 | -0.0029 | 0.0637 | 0.0680 | 0.0145 | -0.0127 | -0.0166 | 0.1494 |
| | RMSE | 0.1849 | 0.1943 | 0.1634 | 0.2458 | 0.1275 | 0.1099 | 0.1204 | 0.2223 |
| 150 | Mean | 1.0012 | 0.9955 | 1.0484 | 1.0486 | 0.5068 | 0.2817 | 0.7881 | 0.8429 |
| | Bias | 0.0012 | -0.0045 | 0.0484 | 0.0486 | 0.0068 | -0.0183 | -0.0119 | 0.1429 |
| | RMSE | 0.1675 | 0.1663 | 0.1611 | 0.2107 | 0.1049 | 0.0961 | 0.1010 | 0.1994 |

**Table 2.** Simulation results with varying values for the four parameters.

| N | Properties | $\alpha = 1.2$ | $\beta = 1.2$ | $\lambda = 1.2$ | $\theta = 1.2$ | $\alpha = 0.9$ | $\beta = 0.9$ | $\lambda = 0.9$ | $\theta = 0.9$ |
|---|---|---|---|---|---|---|---|---|---|
| 20 | Mean | 1.2709 | 1.3037 | 1.2186 | 1.3713 | 0.9496 | 0.9351 | 0.9953 | 1.0418 |
| | Bias | 0.0709 | 0.1037 | 0.0186 | 0.1713 | 0.0496 | 0.0351 | 0.0953 | 0.1418 |
| | RMSE | 0.4045 | 0.2978 | 0.3121 | 0.4545 | 0.3464 | 0.2696 | 0.2671 | 0.3640 |
| 50 | Mean | 1.2216 | 1.2769 | 1.1943 | 1.3150 | 0.9159 | 0.9103 | 0.9703 | 0.9856 |
| | Bias | 0.0216 | 0.0769 | -0.0057 | 0.1150 | 0.0159 | 0.0103 | 0.0703 | 0.0856 |
| | RMSE | 0.2786 | 0.2359 | 0.2577 | 0.3752 | 0.2402 | 0.2244 | 0.2175 | 0.2855 |
| 70 | Mean | 1.2094 | 1.2740 | 1.1878 | 1.3090 | 0.9097 | 0.9051 | 0.9613 | 0.9727 |
| | Bias | 0.0094 | 0.0740 | -0.0122 | 0.1090 | 0.0097 | 0.0051 | 0.0613 | 0.0727 |
| | RMSE | 0.2507 | 0.2200 | 0.2414 | 0.3581 | 0.2158 | 0.2104 | 0.1944 | 0.2605 |
| 100 | Mean | 1.2087 | 1.2558 | 1.1858 | 1.2830 | 0.9041 | 0.9017 | 0.9535 | 0.9612 |
| | Bias | 0.0087 | 0.0558 | -0.0142 | 0.0830 | 0.0041 | 0.0017 | 0.0535 | 0.0612 |
| | RMSE | 0.2179 | 0.1940 | 0.2171 | 0.3162 | 0.1871 | 0.1887 | 0.1741 | 0.2339 |
| 150 | Mean | 1.2027 | 1.2485 | 1.1831 | 1.2693 | 0.9014 | 0.8974 | 0.9466 | 0.9500 |
| | Bias | 0.0027 | 0.0485 | -0.0169 | 0.0693 | 0.0014 | -0.0026 | 0.0466 | 0.0500 |
| | RMSE | 0.1858 | 0.1760 | 0.1861 | 0.2748 | 0.1649 | 0.1684 | 0.1590 | 0.2091 |



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Table 1 and Table 2 presents the results of Monte Carlo simulation carried out to assess the consistency of the parameter estimates. It can be observed that, as sample size increases from 20 to 150 across both tables, there is a consistent decrease in bias and Root Mean Square Error (RMSE), indicating more accurate and precise parameter estimates with larger samples.

**7. Applications**

In this section, we illustrate the flexibility and importance of the EGBXII distribution empirically by two real data applications. The comparison will be made with other existing models which are Burr XII distribution, Sine Burr XII distribution by Isa *et al.* (2022), Transmuted Burr XII distribution by Maurya *et al.* (2017) and Marshal Olkin Gompertz distribution by Eghwerido *et al.* (2021).

The first dataset, discussed by Lawless (1982), represents the number of million revolutions before failure for each of the 23 ball bearings in life tests. The data are as follows:

17.88, 28.92, 33.0, 41.52, 42.12, 45.6, 48.8, 51.84, 51.96, 54.12, 55.56, 67.8, 68.44, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 105.84, 127.92, 128.04, 173.4

**Table 3:** Parameters Estimates and Goodness of Fit Test of the Measures of number of million revolutions before failure of ball bearings in the life tests data set

| MODEL | MLE | -LL | AIC | BIC | CAIC | HQIC | KS | P value |
|---|---|---|---|---|---|---|---|---|
| EGBXII | $\alpha$ =2.4998<br>$\beta$ =0.0014<br>$\lambda$ = 483.74<br>$\theta$ = 1098.4 | 116.7136 | 241.4271 | 245.9691 | 243.6493 | 242.5694 | 0.1270 | 0.8515 |
| BXII | $\alpha$ = 3.7146<br>$\beta$ = 0.0645 | 151.7362 | 307.4724 | 309.7434 | 308.0724 | 308.0435 | 0.5101 | $< 0.05$ |
| SBXII | $\alpha$ = 3.8565<br>$\beta$ = 0.0360 | 150.2546 | 304.5092 | 306.7802 | 305.1092 | 305.0804 | 0.5092 | $< 0.05$ |
| TBXII | $\alpha$ = 0.0026<br>$\beta$ = 60.889<br>$\theta$ = 0.4713 | 153.0819 | 312.1637 | 315.5702 | 313.4269 | 313.0204 | 313.<br>0204 | $< 0.05$ |

Results from Table 3 show that the EGBXII model has the lowest values for AIC, BIC, CAIC, HQIC, and KS, indicating it provides the best fit among the analyzed models. This is further



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supported by a high KS p-value of 0.8515, suggesting a strong fit to the data. The EGBXII model incorporates four parameters, enabling it to effectively capture the behaviour of the dataset compared to competing models with fewer parameters. The EGBXII model's superior performance underscores its effectiveness in modeling failure times in reliability studies, such as the ball bearings dataset. This result holds significant value for industries focused on manufacturing and quality control, as accurate modeling of component failure times aids in predicting maintenance schedules, optimizing resource allocation, and reducing operational costs.

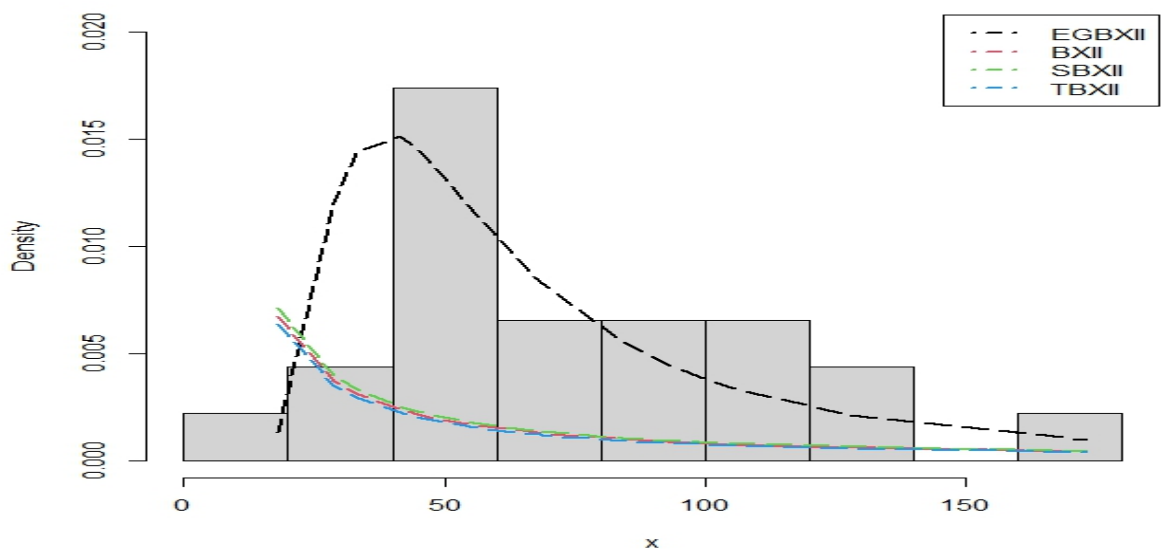


**Figure 5: Fitted pdfs of EGBXII, BXII, SBXII and TBXII distributions on number of million revolutions before failure for each of the 23 ball bearings in the life tests data set**

The fitted PDF plots of the EGBXII, BXII, SBXII, and TBXII distributions for the number of million revolutions before failure of the 23 ball bearings in the life tests dataset, as shown in Figure 5, demonstrate that the EGBXII model fits the dataset more closely than the competing models. This observation is consistent with the findings in Table 3, which indicate that the EGBXII model provides a superior fit.

The second dataset, developed by Reiser *et al.* (1989), contains minimum monthly water flows ($m^3/s$) of the Piracicaba River in São Paulo state, Brazil. The dataset for May is:



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29.19, 8.49, 7.37, 82.93, 44.18, 13.82, 22.28, 28.06, 6.84, 12.14, 153.78, 17.04, 13.47, 15.43, 30.36, 6.91, 22.12, 35.45, 44.66, 95.81, 6.18, 10.00, 58.39, 24.05, 17.03, 38.65, 47.17, 27.99, 11.84, 9.60, 6.72, 13.74, 14.60, 9.65, 10.39, 60.14, 15.51, 14.69, 16.44.

**Table 4:** Parameters Estimates and Goodness of Fit Test of the minimum monthly water flows ($m^3/s$) of the Piracicaba River in São Paulo

| MODEL | MLE | -LL | AIC | BIC | CAIC | HQIC | KS | P value |
|---|---|---|---|---|---|---|---|---|
| EGBXII | $\alpha$ =5.3644<br>$\beta$ =0.0071<br>$\lambda$ = 40.89<br>$\theta$ = 57.29 | 160.8133 | 329.6266 | 336.2808 | 330.803 | 332.0141 | 0.078 | 0.9568 |
| BXII | $\alpha$ =6.1882<br>$\beta$ =0.0540 | 198.4440 | 400.8881 | 404.2152 | 401.2214 | 402.0818 | 0.4558 | $< 0.05$ |
| SBXII | $\alpha$ = 5.4670<br>$\beta$ =0.0352 | 196.1202 | 396.2404 | 399.5675 | 396.5737 | 397.4341 | 0.4483 | $< 0.05$ |
| MOG | $\beta$ = 0.0005<br>$\alpha$ =1.6$e$-05<br>$\lambda$ = 0.0422 | 169.1366 | 344.2732 | 349.2639 | 344.9589 | 346.0638 | 0.1749 | 0.1630 |

Results from Table 4 show that the EGBXII model has the lowest values for AIC, BIC, CAIC, HQIC, and KS, indicating it provides the best fit among the competing models for the minimum monthly water flows ($m^3/s$) dataset. This conclusion is further supported by a high KS p-value of 0.9568, suggesting a strong fit to the data. The EGBXII model incorporates four parameters, which allows it to capture the data's behavior more effectively than the competing models with fewer parameters. The model's ability to provide a superior fit to minimum flow levels carries significant practical implications for ecosystem management, optimizing industrial water usage, and supporting regional planning efforts, especially in regions prone to drought or water shortages



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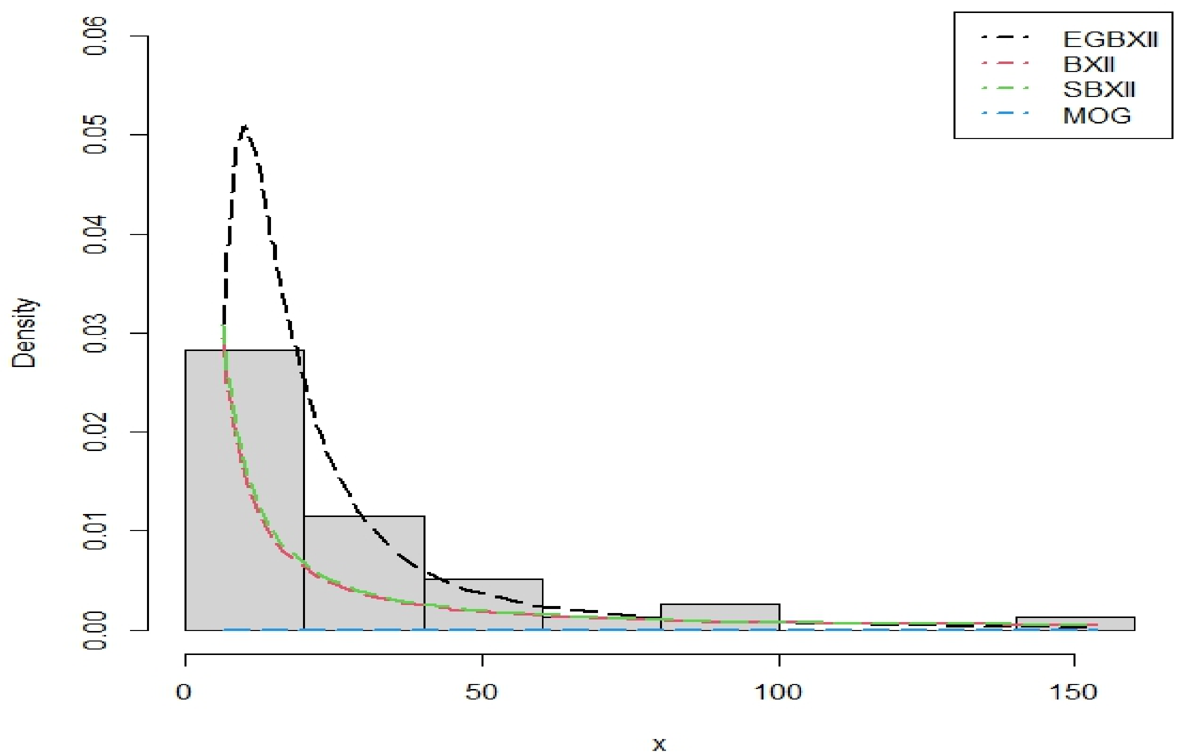


**Figure 6: Fitted pdfs of EGBXII, BXII, SBXII and MOG distributions on minimum monthly water flows ($m^3/s$) data set**

The fitted PDF plots of the EGBXII, BXII, SBXII, and MOG distributions for the minimum monthly water flows ($m^3/s$) dataset, as shown in Figure 6, demonstrate that the EGBXII model fits the dataset more closely than the competing models. This observation is consistent with the results in Table 4, which confirm the superior performance of the EGBXII model.

## 8. Conclusion

This study successfully introduced the Exponentiated Generalized Burr XII (EGBXII) distribution, a novel four-parameter distribution created by compounding the classical Burr XII distribution with the Exponentiated Generalized family. The proposed EGBXII distribution exhibits significant flexibility, effectively accommodating a diverse range of hazard rate patterns, including both increasing and decreasing trends. This adaptability is crucial for accurately modeling real-world phenomena, as many datasets display non-monotonic behavior in their hazard rates. Key properties such as moments, the moment generating function, skewness, and



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kurtosis were derived. The maximum likelihood approach was used for parameter estimation, and a simulation study was conducted to explore the behavior of these parameters. The simulation results indicate that the model demonstrates consistency, as both the bias and root mean square error (RMSE) decrease with increasing sample sizes. Application to two real-world datasets revealed that the Exponentiated Generalized Burr XII distribution outperformed other existing distributions, confirming its effectiveness in fitting complex datasets. This advancement has provided a valuable new tool for statistical modeling, enhancing accuracy and reliability in various applications. Future research should explore other classical methods for estimating the parameters of the model, such as the method of moments, least squares estimation, or Bayesian approaches. Additionally, the EGBXII distribution can be applied to datasets from other fields beyond those considered in this study, broadening its potential applications. However, despite the promising performance of the EGBXII distribution, a notable limitation is its inherent positive skewness. This characteristic may hinder its ability to provide an adequate fit for datasets that exhibit left-skewed behaviour, limiting its flexibility in certain applications.

## References

Abubakar Sadiq, I., Doguwa, S. I., Yahaya, A., & Garba, J. (2023). New Generalized Odd Frechet-G (NGOF-G) Family of Distribution with Statistical Properties and Applications. *UMYU Scientifica*, 2(3), 100 –107.

Alanzi, A. R. A., Imran, M., Tahir, M. H., Chesneau, C., Jamal, F., Shakoor, S., & Sami, W. (2023). Simulation analysis, properties and applications on a new Burr XII model based on the Bell-X functionalities. *AIMS Mathematics*, *8*(3), 6970-7004. https://doi.org/10.3934/math.2023352

Al-Babtain, A. A., Elbatal, I., Al-Mofleh, H., Gemeay, A. M., Afify, A. Z., & Sarg, A. M. (2021). The flexible Burr X-G family: Properties, inference, and applications in engineering science. *Symmetry*, *13*(3), 474. https://doi.org/10.3390/sym13030474

Aldahlan, M., & Afify, A. Z. (2018). The odd exponentiated half-logistic Burr XII distribution. *Pakistan Journal of Statistics and Operational Research*, *14*, 305–317. https://doi.org/10.18187/pjsor.v14i2.2285

Alsadat, N., Nagarjuna, V. B. V., Hassan, A. S., Elgarhy, M., Ahmad, H., & Almetwally, E. M. (2023). Marshall–Olkin Weibull–Burr XII distribution with application to physics data. *AIP Advances*, *13*(9), 095325. https://doi.org/10.1063/5.0172143

Badr, M., & Ijaz, M. (2021). The Exponentiated Exponential Burr XII distribution: Theory and application to lifetime and simulated data. *Plos one*, 16(3), e0248873.

Benchiha, S., Sapkota, L. P., Al Mutairi, A., Kumar, V., Khashab, R. H., Gemeay, A. M., Elgarhy, M., & Nassr, S. G. (2023). *A new sine family of generalized distributions: Statistical inference with applications*. *Mathematical and Computational Applications, 28*(4), 83. https://doi.org/10.3390/mca28040083

Burr, I. W. (1942). Cumulative frequency functions. The Annals of mathematical statistics, 13(2), 215–232.

Cloo, S. K., Brew, L., Nasiru, S., & Odoi, B. (2023). *On the extension of the Burr XII distribution: Applications and regression*. *Computational Journal of Mathematical and Statistical Sciences, 2*(1), 1–30. https://doi.org/10.21608/cjmss.2023.181739.1000

Cordeiro, G. M., Ortega, E. M. M., & da Cunha, D. C. C. (2013). *The exponentiated generalized class of distributions*. *Journal of Data Science, 11*, 1–27.

Eghwerido, J. T., & Agu, F. I. (2021). *The shifted Gompertz-G family of distributions: Properties and applications*. *Mathematica Slovaca, 71*(5), 1291–1308. https://doi.org/10.1515/ms-2021-0053

Eghwerido, J. T., Ogbo, J. O., & Omotoye, A. E. (2021). The Marshall-Olkin Gompertz distribution: Properties and applications. *Statistica*, *81*(2), 183-215. https://doi.org/10.6092/issn.1973-2201/10993

Hussam, E., Sapkota, L. P., & Gemeay, A. M. (2024). Tangent exponential-G family of distributions with applications in medical and engineering. *Alexandria Engineering Journal*, *105*, 181-203. https://doi.org/10.1016/j.aej.2024.06.034

Ieren, T. G., Abdulkadir, S. S., Okolo, A., & Jibasen, D. (2024). A new Fréchet-G family of continuous probability distributions: Special models, properties, simulation and applications. *JRSS-NIG. Group*, *1*(1), 46-71.

Isa, A. M., Ali, B. A., & Zannah, U. (2022). Sine Burr XII distribution: Properties and application to real data sets. *Arid Zone Journal of Basic and Applied Research*, *1*(3), 48-58.

Isa, A. M., Bashiru, S. O., Kaigama, A., & Khalaf, A. A. (2024). Topp-Leone exponentiated Burr XII distribution: Theory and application to real-life data sets. *Iraqi Statisticians Journal*, *1*(1), 63–72. https://doi.org/10.62933/fsf12e55

Isa, A. M., Doguwa, S. I., Alhaji, B. B., & Dikko, H. G. (2023). Sine Type II Topp-Leone G family of probability distribution: Mathematical properties and application. *Arid Zone Journal of Basic and Applied Research*, *2*(4): 124-138

Isa, A. M., Doguwa, S. I., Alhaji, B. B., & Dikko, H. G. (2024). Sine type II Topp-Leone Burr XII distribution: Mathematical properties and application to biomedical and environmental data. *Iraqi Statisticians Journal*, *1*(2), 30-40.

Jamal, F., Chesneau, C., Nasir, M. A., Saboor, A., Altun, E., & Khan, M. A. (2020). On a modified Burr XII distribution having flexible hazard rate shapes. *Mathematica Slovaca*, *70*(1), 193–212. https://doi.org/10.1515/ms-2017-0344

Kajuru, J. Y., Dikko, H. G., Mohammed, A. S., & Fulatan, A. I. (2023). Odd Gompertz-G Family of Distribution, Its Properties and Applications. *Fudma Journal of Sciences*, 7(3), 351-358.

Kajuru, J. Y., Garba, H. D., Suleiman, A. M., & Ibrahim, A. F. (2024). The Generalized Gompertz-G Family of Distributions: Statistical Properties and Applications. *UMYU Scientifica*, *3*(1), 120–128. https://doi.org/10.56919/usci.2431.014

Lawless, J. F. (1982). Statistical models and methods for lifetime data. *John Wiley & Sons*.

Maurya, R.K., Tripathi, Y.M. & Rastogi, M.K. Transmuted Burr XII Distribution. *J Indian Soc Probab Stat* 18, 177–193 (2017). https://doi.org/10.1007/s41096-017-0024-x

Nafo, A. Y., Ocloo, S. K., & Nasiru, S. (2024). New weighted Burr XII distribution: Statistical properties, applications, and regression. *International Journal of Mathematics and Mathematical Sciences*, 2024, Article 4098771. https://doi.org/10.1155/2024/4098771

Ogunde, A. A., & Adeniji, O. E. (2022). Type II Topp-Leone Burr XII distribution: Properties and applications to failure time data. *Scientific African, 16,* e01200. https://doi.org/10.1016/j.sciaf.2022.e01200

Paranaiba, P. F., Ortega, E. M., Cordeiro, G. M., & de Pascoa, M. A. R. (2013). The Kumaraswamy Burr XII distribution: Theory and practice. *Journal of Statistical Computation and Simulation, 83,* 2117–2143. https://doi.org/10.1080/00949655.2012.683003

Qayoom, D., Rather, A. A., Alsadat, N., Hussam, E., & Gemeay, A. M. (2024). A new class of Lindley distribution: System reliability, simulation and applications. *Heliyon*. https://doi.org/10.1016/j.heliyon.2024.e38335

Reiser, D. W., Wesche, T. A., & Estes, C. (1989). Status of instream flow legislation and practices in North America. *Fisheries*, *14*(2), 22–29.

Santos, M. C. T., & Pescim, R. R. (2023). A new extension of the Burr XII distribution generated by odd log-logistic random variables. Communications in Statistics - Theory and Methods, 53(14), 5003–5017. https://doi.org/10.1080/03610926.2023.2200560



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