GARIMA FIRST-ORDER AUTOREGRESSIVE PROCESS: PROPERTIES, ESTIMATION METHODS AND APPLICATION

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ABSTRACT

A new stationary autoregressive process with the Garima marginal distribution is introduced in this paper. Properties of the model such as the distribution of the corresponding error term, conditional moments, time irreversibility, autocorrelation function, spectral density and run probabilities are extensively studied. A simulation study is carried out to compare the performance of the Yule-Walker, conditional least squares and Gaussian estimation procedures in estimating the parameters of the new model. The simulation results indicate that the Gaussian estimation technique is the best among the three methods. The fit of the model to German bilateral real exchange rate data is compared with fits of three existing AR(1) models namely, Gaussian, Exponential and Lindley AR(1) models using Akaike information criterion (AIC) and Bayesian information criterion (BIC). The proposed model is found to be the best for modeling the data among the fitted models since it corresponds to the smallest value of the AIC and BIC.

Keywords: Autocorrelation function, Conditional moments, First-order autoregressive process, Garima distribution, Non-Gaussian marginal distribution, Time irreversibility.

1. Introduction

Observations made at discrete time points, especially regularly space time intervals and their analysis are of necessity in a variety of fields, including finance, medicine, education, agriculture and engineering. Collections of such observation are usually referred to as time series. The first-order autoregressive [AR(1)] process has gained popularity among the probability models for time series analysis. It's application cuts across fields like Regression Analysis (Durbin and Watson, 1950), meteorology, hydrology (Kendall and Dracup, 1991), agriculture (Rai and Satyananda, 2024), among others. The first known AR(1) process is the Gaussian AR(1) process introduced by Yule (1927). The normality assumption of the marginal distribution and the error term corresponding to the model is critical to the application of the process.

Often, time series data with the AR(1) autocorrelation pattern possess characteristics, such as positive skewness and any of platykurticity and leptokurticity which make the assumption of the Gaussian marginal distribution unreasonable for the data. In order to circumvent the limitation of the Gaussian AR(1) process, researchers have cultivated interest in the introduction of AR(1) models with non-Gaussian marginal distributions. In particular, AR(1) processes with the exponential, gamma, inverse Gaussian, Uniform, normal-Laplace, Lindley and double Lindley marginal distributions were proposed by Gaver and Lewis (1980), Abraham and Balakrishna

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(1999), Ristic and Popovic (2000), Jose et al. (2008), Popovic and Bakouch (2016) and Nitha and Krishnarani (2021) respectively. The Garima distribution of Shanker (2016) is one of the notable one-parameter continuous distributions.

The probability density function (pdf) and cumulative distribution function (cdf) of a non-negative continuous random variable X with Garima distribution are defined in (1) and (2) respectively as:

$$f(x) = \left(\frac{\lambda}{\lambda+2}\right)(1+\lambda+\lambda x)e^{-\lambda x} \qquad x>0;\ \lambda>0 \tag{1}$$

$$F(x) = 1-\left[1+\frac{\lambda x}{\lambda+2}\right]e^{-\lambda x} \qquad x>0;\ \lambda>0 \tag{2}$$

The mean (E(X)) and variance (Var(X)) of this distribution respectively are

$E(X) = \dfrac{(\lambda+3)}{\lambda(\lambda+2)}$ and $Var(X) = \dfrac{\lambda^2+6\lambda+7}{\lambda^2(\lambda+2)^2}$

As a one-parameter distribution, the Garima distribution has been empirically established as being capable of outperforming one-parameter distributions such as the exponential, Lindley and Shanker distributions when they are fitted to some real life data sets (Shanker, 2016). In spite of this interesting property of the distribution, its performance when it is used as a marginal distribution to construct an AR(1) model have not been studied. Therefore, the principal intention of writing this scholarly work is to propose a new AR(1) process with the Garima marginal distribution.

2. Methods

Here, attention is given to the definition of the new process, its corresponding mathematical characteristics, and estimation of the method.

2.1 Definition of the Garima First-Order Autoregressive GaAR(1) Process [GaAR(1)] Process and Distribution of its Related Innovation Sequence

Consider the first-order stationary autoregressive process

$$X_t = \alpha X_{t-1} + e_t \qquad \alpha \in (0,\ 1) \tag{3}$$

Equation (3) is called the first-order stationary autoregressive process with Garima marginal distribution, denoted as GaAR(1) process if $\{X_t\}$ is a stationary process having the Garima marginal distribution and $\{e_t\}$ refers to a sequence of i.i.d random variables independent of X_{t-k} for $k \geq 1$. To derive the distribution of $\{e_t\}$, we first of all determine the Laplace transform of the Garima distribution with parameter λ [Ga(λ) distribution]. The Laplace transform of the distribution is defined as

$$\phi_X(s) = \frac{\lambda}{\lambda+2}\int_0^{\infty}(1+\lambda+\lambda x)e^{-(\lambda+s)x}dx$$

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$$=\left(\frac{\lambda}{\lambda+2}\right)\left(\frac{(\lambda+1)(\lambda+s)+\lambda}{(\lambda+s)^2}\right) \tag{4}$$

Consequently,

$$\phi_X(\alpha s)=\left(\frac{\lambda}{\lambda+2}\right)\left(\frac{(\lambda+1)(\lambda+\alpha s)+\lambda}{(\lambda+\alpha s)^2}\right) \tag{5}$$

Lemma 1. Let

$$A_1=\frac{\left((\lambda+1)(1-\alpha)^2+2(1-\alpha)\right)\lambda+2}{\left((\lambda+1)(1-\alpha)+1\right)^2}\ ,\ B_1=\frac{1-\alpha}{(\lambda+1)(1-\alpha)+1}\ ,\ \text{and}\ C_1=\frac{\alpha}{\left((\lambda+1)(1-\alpha)+1\right)^2}$$

If $\lambda>0$, $x>0$ and $|\alpha|<1$, the function

$$g(x)=A_1\lambda e^{-\theta x}+B_1\lambda^2 xe^{-\theta x}-C_1\,\tfrac{\lambda(\lambda+2)}{\alpha(\lambda+1)}\,e^{-\left[\frac{(\lambda+2)}{\alpha(\lambda+1)}\right]\lambda x}$$

is a proper probability density function (pdf).

Proof. To proof that g(x) is a density function, we need to establish that $\int_0^\infty g(x)\,dx=1$ and $g(x)\geq 0$. Since g(x) is a mixture of Exponential(θ), Gamma($2,\theta$) and Exponential$\left[\dfrac{\theta(\theta+2)}{\alpha(\theta+1)}\right]$, it follows that

$$\int_0^\infty g(x)\,dx=\int_0^\infty\left(A_1\lambda e^{-\theta x}+B_1\lambda^2 xe^{-\theta x}-C_1\,\tfrac{\lambda(\lambda+2)}{\alpha(\lambda+1)}\,e^{-\left[\frac{(\lambda+2)}{\alpha(\lambda+1)}\right]\lambda x}\right)dx$$

$$=A_1+B_1-C_1=1$$

An alternative representation of g(x) is

$$g(x)=\frac{\lambda e^{-\lambda x}}{(\lambda+1)(1-\alpha)+1}\,g_1(x)\,,$$

where,

$$g_1(x)=\frac{\lambda\left[(\lambda+1)(1-\alpha)^2+2(1-\alpha)\right]+2}{\left[(\lambda+1)(1-\alpha)+1\right]}+(1-\alpha)\lambda x-\frac{\alpha(\lambda+2)}{\alpha(\lambda+1)\left[(\lambda+1)(1-\alpha)+1\right]}e^{-\left(\frac{(\lambda+2)}{\alpha(\lambda+1)}-1\right)\lambda x}$$

Now,

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$$g_1(0) = \frac{\lambda\left((\lambda+1)(1-\alpha)^2 + 2(1-\alpha)\right) + 2}{(\lambda+1)\left[(\lambda+1)(1-\alpha)+1\right]} > 0$$

Again,

$$g_1'(x) = \frac{dg_1(x)}{dx} \;=\; (1-\alpha)\lambda + \frac{\lambda(\lambda+2)}{\alpha(\lambda+1)^2} e^{-\left(\frac{\lambda+2}{\alpha(\lambda+1)}-1\right)\lambda x},$$

which is greater than 0 for $x > 0$, $\lambda > 0$ and $|\alpha| < 1$

Additionally, $\lim_{x\to\infty} g_1(x) = \infty$.From the foregoing, it is clear that g(x) satisfies the assumption of the pdf of a continuous random variable.

Theorem 1: Suppose the marginal distribution for the AR(1) process in Equation (3) is the Garima distribution with parameter λ. The innovation sequence $\{e_t\}$ has a distribution which is a mixture of the singular distribution and absolute continuous distributions defined by

$$f_e(x) = \alpha\delta(x) + (1-\alpha)g(x), \tag{6}$$

where,

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

is the Dirac delta function.

Proof. Since X_t is a stationary process, if we take into consideration the properties of the innovation sequence, then the Laplace transform (LT) of the sequence can be deduced from Equation (3) as

$$\phi_e(s) = \frac{\phi_X(s)}{\phi_X(\alpha s)} \tag{7}$$

Applying Equations (4) and Equation (5), in Equation (7) gives

$$\phi_e(s) = \frac{(\lambda+\alpha s)^2\left[(\lambda+1)(\lambda+s)+\lambda\right]}{(\lambda+s)^2\left[(\lambda+1)(\lambda+\alpha s)+\lambda\right]} \tag{8}$$

Equivalently,

$$\phi_e(s) = \alpha + (1-\alpha)\frac{\left(\alpha\lambda^2 s^2 + \lambda^2(1+\alpha)(\lambda+1)s + \lambda^4 + 2\lambda^3\right)}{(\lambda+s)^2\left((\lambda+1)(\lambda+\alpha s)+\lambda\right)} \tag{9}$$

By partial fraction decomposition, we obtain

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$$\frac{\alpha\lambda^2 s^2 + \lambda^2(1+\alpha)(\lambda+1)s + \lambda^4 + 2\lambda^3}{(\lambda+s)^2\left((\lambda+1)(\lambda+\alpha s)+\lambda\right)} = \frac{A}{\lambda+s} + \frac{B}{(\lambda+s)^2} + \frac{C}{(\lambda+1)(\lambda+\alpha s)+\lambda}$$

Solving for A, B and C leads to

$$A = \frac{\left((\lambda+1)(1-\alpha)^2 + 2(1-\alpha)\right)\lambda^2 + 2\lambda}{\left((\lambda+1)(1-\alpha)+1\right)^2}, \quad B = \frac{\lambda^2(1-\alpha)}{(\lambda+1)(1-\alpha)+1} \quad \text{and} \quad C = \frac{-\alpha\lambda(\lambda+2)}{\left((\lambda+1)(1-\alpha)+1\right)^2}$$

Substituting the expression for A, B and C into Equation 8 yields

$$\phi_e(s) = \alpha + (1-\alpha)\left(A_1\left(\frac{\lambda}{\lambda+s}\right) + B_1\left(\frac{\lambda}{\lambda+s}\right)^2 - C_1\left(\frac{\lambda(\lambda+2)}{(\lambda+1)(\lambda+\alpha s)+\lambda}\right)\right)$$

Hence, $\phi_e(s)$ is a mixture of discrete component with probability α and Exponential(λ), Gamma(2, λ) and Exponential$\left(\frac{\lambda(\lambda+2)}{\alpha(\lambda+1)}\right)$ with probability $1-\alpha$.

2.2 Conditional Moments

Given the GaAR(1) process, the one-step ahead conditional mean is

$$E(X_t / X_{t-1} = x_{t-1}) = \alpha x_{t-1} + E(e_t)$$

For the two-step ahead conditional mean, we have

$$E(X_{t+1} / X_{t-1} = x_{t-1}) = \alpha E(X_{t+1} / X_{t-1} = x_{t-1}) + E(e_t)$$

$$= \alpha^2 x_{t-1} + \alpha E(e_t) + E(e_t)$$

In general, the $(k+1)^{th}$ step ahead conditional mean is

$$E(X_{t+k} / X_{t-1} = x_{t-1}) = \alpha^{k+1} x_{t-1} + (1+\alpha+\alpha^2+\ldots+\alpha^{k-1}+\alpha^k)E(e_t)$$

$$= \alpha^{k+1} x_{t-1} + \frac{(1-\alpha^{k+1})}{(1-\alpha)} E(e_t)$$

$$= \alpha^{k+1} x_{t-1} + (1-\alpha^{k+1})E(X_t)$$

$$= \alpha^{k+1} x_{t-1} + \frac{(1-\alpha^{k+1})(\lambda+3)}{\lambda(\lambda+2)}$$

As $k \to \infty$, $E(X_{t+k} / X_{t-1} = x_{t-1}) \to \frac{(\lambda+3)}{\lambda(\lambda+2)}$ which is the unconditional mean of the GaAR(1) process.

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Furthermore, the one-step ahead conditional variance is

$$Var(X_t / X_{t-1} = x_{t-1}) = \alpha^2 Var(X_{t-1} / X_{t-1} = x_{t-1}) + Var(e_t / X_{t-1} = x_{t-1})$$

$$= Var(e_t)$$

In the case of the two-step ahead conditional variance, we have

$$Var(X_{t+1} / X_{t-1} = x_{t-1}) = \alpha^2 Var(X_t / X_{t-1} = x_{t-1}) + Var(e_t / X_{t-1} = x_{t-1})$$

$$= (\alpha^2 + 1) Var(e_t)$$

Thus, k+1 ahead conditional variance In general,

$$Var(X_{t+k} / X_{t-1} = x_{t-1}) = (\alpha^{2k} + \alpha^{2k-2} + \cdots + \alpha^2 + 1) Var(e_t)$$

$$= \left(\frac{1-\alpha^{2(k+1)}}{1-\alpha^2}\right)(1-\alpha^2) Var(X_t)$$

$$= (1-\alpha^{2(k+1)})\left(\frac{\lambda^2 + 6\lambda + 7}{\lambda^2 (\lambda+2)^2}\right)$$

Notably, if $k \to \infty$, then $Var(X_{t+k} / X_{t-1} = x_{t-1}) \to \left[\frac{\lambda^2 + 6\lambda + 7}{\lambda^2(\lambda+2)^2}\right]$, which is the unconditional variance of the GaAR(1) process.

2.3 Conditional Laplace Transform, Joint Laplace Transform and Time Irreversibility

Let $\phi_{X_{t+k} / X_{t-1} = x_{t-1}}(s)$ denote the conditional Laplace transform of the GaAR(1) process. Then

$$\phi_{X_{t+k} / X_{t-1} = x_{t-1}}(s) = e^{-s\alpha^{k+1} x_{t-1}} \phi_e\left(\frac{1-\alpha^{k+1}}{1-\alpha} s\right) \quad (10)$$

Applying Equation (8) in Equation 10, and simplifying the result, the following result is obtained

$$\phi_{X_{t+k} / X_{t-1} = x_{t-1}}(s) =$$

$$e^{-s\alpha^{k+1} x_{t-1}} \left(\frac{\left[(1-\alpha)\lambda + \alpha(1-\alpha^{k+1})s\right]^2 \left[(\lambda+1)\left((1-\alpha)\lambda + (1-\alpha^{k+1})s\right) + (1-\alpha)\lambda\right]}{\left[(1-\alpha)\lambda + (1-\alpha^{k+1})s\right]^2 \left[(\lambda+1)\left((1-\alpha)\lambda + \alpha(1-\alpha^{k+1})s\right) + (1-\alpha)\lambda\right]}\right) \quad (11)$$

As $k \to \infty$, we have

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$$\phi_{X_{t+k}/X_{t-1}=x_{t-1}}(s) \to \frac{[(1-\alpha)\lambda+\alpha s]^2\left[(\lambda+1)\big((1-\alpha)\lambda+s\big)+(1-\alpha)\lambda\right]}{[(1-\alpha)\lambda+s]^2\left[(\lambda+1)\big((1-\alpha)\lambda+\alpha s\big)+(1-\alpha)\lambda\right]} \quad (12)$$

Specifically, if $\alpha = 0$, we write

$$\phi_{X_{t+k}/X_{t-1}=x_{t-1}}(s) = \frac{\lambda\left[(\lambda+1)(\lambda+s)+\lambda\right]}{(\lambda+2)(\lambda+s)^2}$$

which is the Laplace transform of the Garima distribution with parameter θ

The joint Laplace transform for the GaAR(1) model is given as

$$\phi_{X_{t-1},/X_t}(s_1,s_2) = \phi_{X_{t-1}}(s_1+s_2\alpha)\phi_e(s_2)$$

$$= \left(\frac{\lambda}{\lambda+2}\right)\left[\frac{(\lambda+1)(\lambda+s_1+s_2\alpha)+\lambda}{(\lambda+s_1+s_2\alpha)^2}\right]\times\frac{(\lambda+\alpha s_2)^2\left[(\lambda+1)(\lambda+s_2)+\lambda\right]}{(\lambda+s_2)^2\left[(\lambda+1)(\lambda+\alpha s_2)+\lambda\right]}$$

$$= \frac{\lambda\left[(\lambda+1)(\lambda+s_1+s_2\alpha)+\lambda\right](\lambda+\alpha s_2)^2\left[(\lambda+1)(\lambda+s_2)+\lambda\right]}{(\lambda+2)(\lambda+s_1+s_2\alpha)^2(\lambda+s_2)^2\left[(\lambda+1)(\lambda+\alpha s_2)+\lambda\right]}$$

$\phi_{X_{t-1},/X_t}(s_1,s_2)$ is remarkably not symmetric with regard to s_1 and s_2. As consequence, GaAR(1) model is not time reversible. It is easy to verify that for the GaAR(1) model $\phi_{X_{t-1},/X_t}(0,0) = 1$.

2.4 Autocorrelation and Spectral Density Functions for the Stationary GaAR(1) Process

Let γ_k denote the autocovariance at lag k for the stationary GaAR(1) process.
To derive an appropriate expression for γ_k, we proceed as follows:

If k = 1, we have

$$\gamma_1 = E(X_t, X_{t-1}) = E(X_t)E(X_{t-1})$$

$$= \alpha E(X_{t-1}^2) - E(X_t)E(X_{t-1}) + E(e_t)E(X_{t-1})$$

$$= \alpha\,\mathrm{var}(X_t) = \alpha\,\gamma_0$$

Given that k = 2,

$$\gamma_2 = E(X_t, X_{t-1}) - \left[E(X_t)\right]^2$$

$$= \alpha E(X_{t-1}, X_{t-2}) - \left[E(X_t)\right]^2 + E(e_t)E(X_t)$$

$$= \alpha\,\gamma_1 = \alpha^2\,\gamma_0$$

Therefore,

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$$\gamma_k = Cov(X_t, X_{t-k}) = E(X_t, X_{t-1}) - [E(X_t)]^2$$

$$= \alpha^{|k|}\,\gamma_0$$

$$= \alpha^{|k|}\,\frac{\lambda^2 + 6\lambda + 7}{\lambda^2(\lambda+2)^2} \qquad k = 0, \pm 1, \pm 2, \cdots$$

Symbolically, the autocorrelation at lag k for GaAR(1) process is

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \alpha^{|k|}, \qquad k = 0, \pm 1, \pm 2, \cdots \tag{13}$$

It can be pointed out that the autocorrelation function [acf] of Equation(13) mimics that of the Gaussian AR(1) process in terms of the exponential decay pattern. The spectral density corresponding to the derived acf has the form

$$f(w) = \frac{R(0)}{2\pi}\sum_{k=-\infty}^{\infty}\rho_k e^{-iwk}$$

$$= \frac{R(0)}{2\pi}\sum_{k=-\infty}^{\infty}\alpha^{|k|}e^{-iwk}$$

$$= \frac{(1-\alpha^2)(\theta^2 + 6\theta + 7)}{2\pi(1 - 2\alpha\cos w + \alpha^2)\theta^2(\theta+2)^2} \qquad w \in (-\pi, \pi) \tag{14}$$

2.5 Computation of Run Probabilities

The derivation of the run probability for the GaAR(1) process is the basic task performed in this section. Theorem 1 suggests that the GaAR(1) process can be written in the form:

$$X_t = \begin{cases} \alpha X_{t-1}, & w.p\ \alpha \\ \alpha X_{t-1} + e_t, & w.p\ (1-\alpha) \end{cases} \tag{15}$$

where, w.p stands for “with probability”.

Hence, the probability of runs for the new process is

$$P(X_t < X_{t-1}) = \alpha P(\alpha X_{t-1} < X_{t-1}) + (1-\alpha)P(e_t < (1-\alpha)X_{t-1})$$

$$= \alpha + (1-\alpha)P(e_t < (1-\alpha)X_{t-1}) \tag{16}$$

Notably,

$$P(e_t < (1-\alpha)X_{t-1}) = \int_0^{\infty} F\left(\frac{x}{1-\alpha}\right) g(x)\,dx,$$

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where F(.) represents the cdf of the Garima distribution and g(.) as defined in Lemma 1. Let

$$I_1 = \left(1 - \left(\frac{\lambda + 2 + \lambda(\frac{x}{1-\alpha})}{\lambda + 2}\right) e^{-\frac{\lambda x}{1-\alpha}}\right) \text{ and } I_2 = \left(A_1 \lambda e^{-\theta x} + B_1 \lambda^2 x e^{-\theta x} - \frac{C_1 \lambda(\lambda+2)}{\alpha(\lambda+1)} e^{-\left[\frac{\lambda(\lambda+2)}{\alpha(\lambda+1)}\right]\lambda x}\right)$$

$$P(e_t < (1-\alpha)X_{t-1}) = \int_0^\infty I_1 I_2 \, dx,$$

$$= 1 - \left(\frac{1-\alpha}{\lambda+2}\right)\left(F_1 + F_2 - F_3\right) \tag{17}$$

where, $F_1 = A_1 \frac{[(\lambda+2)(2-\alpha)+1]}{(2-\alpha)^2}$, $F_2 = B_1(1-\alpha)\left[\frac{(\lambda+2)(2-\alpha)+2}{(2-\alpha)^3}\right]$ and

$$F_3 = C_1 \frac{(\lambda+2)[(\lambda+2)(\lambda+2-\alpha)+\alpha(\lambda+1)]}{(\lambda+2-\alpha)^2}$$

Substituting Equation (17) into Equation (16) gives the result

$$\therefore\ P(X_t < X_{t-1}) = 1 - \frac{(1-\alpha)^2}{\lambda+2}\left(F_1 + F_2 - F_3\right) \tag{18}$$

Run probabilities which correspond to various combinations of α = 0.5, 1, 3 and λ = 0.1 to 0.8 are contained in Table 1.

Table 1: Run Probabilities for the GaAR(1) process based on the selected values of the associated parameters.

	α								
λ	0.1	0.2	0.3	0.4	0.5	0.5	0.6	0.7	0.8
0.5	0.578	0.6527	0.7235	0.7893	0.8491	0.9012	0.944	0.9756	0.9943
1	0.5768	0.6504	0.7201	0.7851	0.8442	0.8962	0.9396	0.9725	0.9931
3	0.5749	0.6467	0.7149	0.7787	0.8372	0.8893	0.9337	0.9686	0.9916

Table 1 reveals that if λ is fixed, the run probability increases as the value of α increases, and by holding α constant and increasing the value of λ, results in the decrease of the value of the probability. This implies that the proposed model is relatively more flexible than the well-known Gaussian AR (1) counterpart.

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2.6 Estimation methods

Three estimation procedures, namely, the Yule-Walker, conditional least squares and Gaussian estimation techniques have been adopted in this study for the purpose of estimating the new model parameters. They are extensively explained below.

(a) Yule-Walker method

Let $\hat{\alpha}_{YW}$ and $\hat{\lambda}_{YW}$ represent the Yule-Walker (YW) estimators of α and λ respectively. Equating the theoretical autocorrelation at lag 1 to its sample counterparts gives

$$\hat{\alpha}_{YW} = \frac{\sum_{t=2}^{n}(X_t - \bar{X})(X_{t-1} - \bar{X})}{\sum_{t=1}^{n}(X_t - \bar{X})^2} \tag{19}$$

If we equate the process mean and the sample mean, then

$$\frac{\hat{\lambda}_{YW} + 3}{\hat{\lambda}_{YW}(\hat{\lambda}_{YW} + 2)} = \bar{X} \tag{20}$$

$$\hat{\lambda}_{YW} = \frac{-(2\bar{X} - 1) + \sqrt{(2\bar{X} - 1)^2 + 12\bar{X}}}{2\bar{X}} \tag{21}$$

Properties of the YW estimators for AR(1) processes are well-known and can be found in Stoica et al. (1989), Basu and Reinsel, (1992) and among others.

(b) Conditional least squares methodology

The conditional least squares estimators of α and λ, which are respectively denoted by $\hat{\alpha}_{CLS}$ and $\hat{\lambda}_{CLS}$ are the estimators that minimize

$$Q^* = \sum_{t=2}^{n}(X_t - E(X_t / X_{t-1}))^2$$

$$= \sum_{t=2}^{n}(X_t - \alpha X_{t-1} - (1-\alpha)E(X_t))^2$$

Let $\mu = E(X_t)$.

Then for $\frac{\partial Q^*}{\partial \alpha} = 0$

$$\sum_{t=2}^{n} X_{t-1}X_t - \alpha\sum_{t=2}^{n} X_{t-1}^2 - (1-\alpha)\mu\sum_{t=2}^{n} X_{t-1} - \mu\sum_{t=2}^{n} X_t + \alpha\mu\sum_{t=2}^{n} X_{t-1} + (n-1)(1-\alpha)\mu^2 = 0 \tag{22}$$

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Again if $\frac{\partial Q^*}{\partial \mu} = 0$

$$\sum_{t=2}^{n} X_t - \alpha \sum_{t=2}^{n} X_{t-1} - (n-1)(1-\alpha)\mu = 0 \tag{23}$$

Using Equation (23), we have

$$\hat{\mu} = \frac{\sum_{t=2}^{n} X_t - \alpha \sum_{t=2}^{n-2} X_{t-1}}{(n-1)(1-\alpha)} \tag{24}$$

Applying Equation (23) and Equation (24) in Equation (22) yields

$$\hat{\alpha}_{CLS} = \frac{(n-1)\sum_{t=2}^{n} X_{t-1}X_t - \sum_{t=2}^{n} X_{t-1} \sum_{t=2}^{n} X_t}{(n-1)\sum_{t=2}^{n} X^2_{t-1} - \left(\sum_{t=2}^{n} X_{t-1}\right)^2} \tag{25}$$

Replacing α in equation (24) by its CLS estimator $\hat{\alpha}_{CLS}$ leads to the CLS estimator of μ as

$$\hat{\mu}_{CLS} = \frac{\sum_{t=2}^{n} X_t - \hat{\alpha}_{CLS} \sum_{t=2}^{n-2} X_{t-1}}{(n-1)(1-\hat{\alpha}_{CLS})} \tag{26}$$

This implies that

$$\hat{\mu}_{CLS} = \frac{\hat{\lambda}_{CLS} + 3}{\hat{\lambda}_{CLS}(\hat{\lambda}_{CLS} + 2)} \tag{27}$$

Using Equation (27), we have

$$\hat{\theta}_{CLS} = \frac{-(2\hat{\mu}_{CLS} - 1) + \sqrt{(2\hat{\mu}_{CLS} - 1)^2 + 12\hat{\mu}_{CLS}}}{2\hat{\mu}_{CLS}} \tag{28}$$

(c) Gaussian Estimation Procedure

Gaussian estimation due has recently been accepted as one of the approaches to estimating parameters of non-Gaussian AR (1) process. We commence our discussion of this method by considering the conditional likelihood function

$$L(\alpha, \theta) = f(x_1)\prod_{t=2}^{n} f(x_t / x_{t-1}), \tag{29}$$

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where, $f(x_1)$ and $f(x_t / x_{t-1})$ are the marginal probability function of $\{X_t\}$ and related conditional probability function. Suppose that $m_{x_{t-1}} = E(X_t / X_{t-1} = x_{t-1})$ and $\sigma^2_{x_{t-1}} = Var(X_t / X_{t-1} = x_{t-1})$. The assumption of normal distribution for both of $f(x_1)$ and $f(x_t / x_{t-1})$ implies that the log-likelihood function for the GaAR(1) process can be stated as

$$\ell = \ln(L(\alpha,\theta)) == n\ln\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum_{t=2}^{n}\left(\ln\sigma^2_{x_{t-1}} + \frac{(x_t - m_{x_{t-1}})^2}{\sigma^2_{x_{t-1}}}\right) \tag{30}$$

In the case of the GaAR (1) process,

$$m_{x_{t-1}} = \alpha\, x_{t-1} + \frac{(1-\alpha)(\lambda+3)}{\lambda(\lambda+2)} \text{ and, } \sigma^2_{x_{t-1}} = (1-\alpha^2)\left[\frac{\lambda^2 + 6\lambda + 7}{\lambda^2(\lambda+2)^2}\right]$$

$$\ell == n\ln\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{1}{2}\sum_{t=2}^{n}\left(\ln\left[\frac{(1-\alpha^2)\lambda^2 + 6\lambda + 7}{\lambda^2(\lambda+2)^2}\right] + \frac{\lambda^2(\lambda+2)^2\left(x_t - \alpha\, x_{t-1} + \frac{(1-\alpha)(\lambda+3)}{\lambda(\lambda+2)}\right)^2}{\frac{(1-\alpha^2)\lambda^2 + 6\lambda + 7}{\lambda^2(\lambda+2)^2}}\right)$$

Gaussian estimates of α and λ are determined by solving $\frac{\partial\ell}{\partial\alpha} = 0$ and $\frac{\partial\ell}{\partial\lambda} = 0$ simultaneously.

Let $\hat{\alpha}_{GE}$ and $\hat{\lambda}_{GE}$ be the Gaussian estimators of α and λ respectively. Under established regularity conditions, the asymptotic distribution of $\sqrt{n}\left((\hat{\alpha}_{Ge}, \hat{\lambda}_{GE})' - (\alpha, \lambda)'\right)$ is bivariate normal with mean vector $(0,0)'$ and covariance matrix $(I(\alpha, \lambda))^{-1}$, where

$$I(\alpha,\theta) = -E\begin{bmatrix} \frac{\partial^2\ell}{\partial\alpha^2} & \frac{\partial^2\ell}{\partial\alpha\partial\theta} \\ \frac{\partial^2\ell}{\partial\theta\partial\alpha} & \frac{\partial^2\ell}{\partial\alpha^2} \end{bmatrix}$$

Consequently, the maximum likelihood estimator $\hat{\Theta}_{GE} = (\hat{\alpha}_{GE}, \overleftarrow{\lambda}_{GE})'$ of $\Theta_{GE} = (\alpha, \lambda)'$ is consistent.

3. Results and Discussion

3.1 Simulation study on GaAR (1) process

In this section, a simulation study is given due consideration for the purpose of comparing the Yule-walker (YW), conditional least squares (CLS) and Gaussian estimation (GE) methods of obtaining estimates of the GaAR (1) model.

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This was done by simulating 1000 replicates of different samples of sizes n = 100, 250, 500 and 1000 from GaAR (1) process for the parameter values (a) α = 0.5, ƛ= 0.5, and (b) α = 0.5, θ = 1.0. The estimates obtained are presented in Tables 2.

Table 2: GaAR (1) Simulation result

		α = 0.5, ƛ= 0.5				**α = 0.5, θ = 1.0**			
Method		100	250	500	1000	100	250	500	1000
YW	$\hat{\alpha}$	0.4655	0.4883	0.4923	0.497	0.4737	0.4837	0.4902	0.4973
	Bias($\hat{\alpha}$)	-0.0345	-0.0116	-0.0075	-0.003	-0.0263	-0.0163	-0.0098	-0.0027
	MSE($\hat{\alpha}$)	0.0067	0.0029	0.0015	0.0007	0.0076	0.0029	0.0016	0.0008
	$\hat{\lambda}$	0.5153	0.5024	0.5038	0.5015	1.055	1.0092	1.003	1.0052
	Bias($\hat{\lambda}$)	0.0153	0.0025	0.0038	0.0015	0.055	0.0092	0.0033	0.0052
	MSE($\hat{\lambda}$)	0.0078	0.0022	0.0013	0.0006	0.0334	0.0092	0.0047	0.0022
CLS	$\hat{\alpha}$	0.469	0.4908	0.4936	0.4975	0.4779	0.4853	0.4913	0.4978
	Bias($\hat{\alpha}$)	-0.031	-0.0092	-0.0064	-0.0025	-0.0221	-0.0147	-0.0087	-0.0022
	MSE($\hat{\alpha}$)	0.0066	0.0029	0.0149	0.0007	0.0075	0.0028	0.0016	0.0008
	$\hat{\lambda}$	0.506	0.4986	0.5019	0.5006	1.0367	1.0021	0.9995	1.0033
	Bias($\hat{\lambda}$)	0.006	-0.0014	0.0019	0.0006	0.0367	0.0021	-0.0005	0.0033
	MSE($\hat{\lambda}$)	0.0071	0.0022	0.0334	0.0006	0.0307	0.0091	0.0046	0.0022
GE	$\hat{\alpha}$	0.4785	0.4958	0.4955	0.4988	0.4867	0.4899	0.4943	0.4989
	Bias($\hat{\alpha}$)	-0.0215	-0.0042	-0.0045	-0.0117	-0.0133	-0.0101	-0.0057	-0.0011
	MSE($\hat{\alpha}$)	0.0052	0.0023	0.0013	0.0006	0.006	0.0024	0.0013	0.0007
	$\hat{\lambda}$	0.5112	0.5037	0.5031	0.5019	1.049	1.011	1.007	1.0052
	Bias($\hat{\lambda}$)	0.0112	0.0037	0.0031	0.0019	0.049	0.011	0.007	0.0052
	MSE($\hat{\lambda}$)	0.0061	0.0022	0.001	0.0005	0.0295	0.0088	0.0045	0.0019

Results contained in Table 2 indicate that the average biases and mean squares decrease as the sample size increases. The GE seem to approach the parameter values faster as the sample size increases compared to other estimation methods, and the mean squared error pertaining to the GE method appeared to be the smallest in each case. Therefore, on balance, the GE method is the best for estimating the parameter of the GaAR (1) process.

3.2 Application to real data

The applicability of the proposed process is established by using German bilateral real exchange rate GBREX (Average exchange rate) data sourced from External Sector Statistics Central Bank of Nigeria Statistical Bulletin from January 2008 to December 2023 (http//:www.cbn.org.ng).

We presented the descriptive statistics of the data on Table 3. The result shows that the Skewness and kurtosis of the data is 0.332 and -0.793 respectively, and therefore can be modeled using AR (1) processes having non-Gaussian marginal distribution.

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Table 3: The descriptive statistics of the data

Mean	Median	Stdev	Variance	Skewness	Kurtosis	Max	Min
166.24	158.79	27.82	773.92	0.351	-0.687	226.52	117.69

Furthermore, the ACF and PACF of the data were determined and the resultant ACF as seen in Figure 1 decays exponentially and the PACF cuts of after lag 1. Therefore, we modeled the data using the non-Gaussian stationary AR(1) processes.

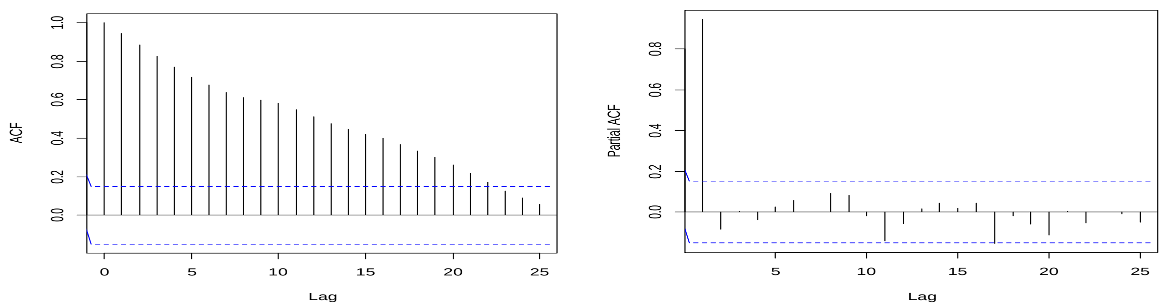

Figure 1: ACF and PACF of the GBREX data

Next, the GE method was applied to estimate the parameter of the process. To ensure that the right model was fitted, we compared GaAR(1) process with the Gaussian AR(1) process, the exponential AR(1) process and the Lindley AR(1) process. Thus, the same data was used to fit The GaAR(1), EAR(1), LAR(1) and AR(1) processes, and the best selected using the negative log-likelihood, Akaike information criterion (AIC) and Bayesian information criterion (BIC) values. The values used as presented on Table 4 convey that GaAR(1) model favourably competes with other models with all the model selection criteria lesser than others. Hence, we conclude that the GaAR(1) is better than others in modeling the data under consideration.

Table 4: GaAR(1), EAR(1), LAR(1) and AR(1) models with their AIC and BIC values

Model	α	λ	$-\ell$	AIC	BIC
GaAR(1)	0.9930	0.0210	583.25	1170.50	1177.02
EAR(1)	0.9934	0.0154	583.29	1170.57	1177.09
LAR(1)	0.9983	-0.0111	583.67	1171.33	1177.85
AR(1)	0.9668	---	586.88	1179.76	1189.18

Finally, the residual diagnostic performed show that the ACF and PACF of the residuals are within the limits as expected, and hence random (see Figure 2).

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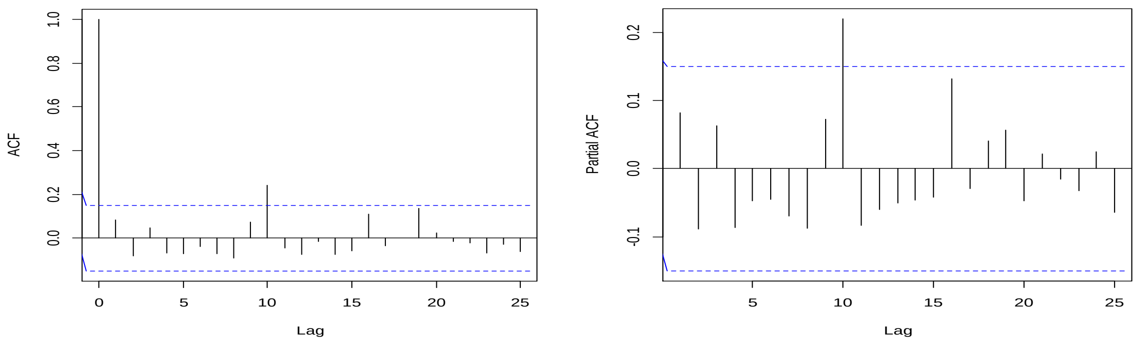

Figure 2: Residual ACF and PACF

4. Conclusion

In this scholarly work, we have introduced a new stationary AR(1) model with the Garima distribution marginal. The constructed model possesses time irreversibility property. Its associated innovation distribution was established to be a mixture of the singular and absolute continuous distributions. Run probabilities corresponding to the model increase as the value of one of the model parameters increases while the other parameter is held constant. Comparison of the methods of estimating the parameters of the model via a simulation study and each of average bias and mean squared error reveals that the Gaussian estimation method is the best for estimating the parameters compared to the Yule-Walker and conditional least squares method. The GaAR(1) model competes favourably well for German bilateral real exchange rate data (series) relative to the Gaussian AR(1), exponential AR(1) and Lindley AR(1) models.

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