A TYPE I HALF LOGISTIC TOPP-LEONE INVERSE WEIBULL DISTRIBUTION: STATISTICAL PROPERTIES AND APPLICATIONS

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ABSTRACT

Researchers in the area of statistical distribution have made efforts on generalizing the existing probability distributions to improve their modeling flexibility by adding extra parameters. In this paper, a new continuous probability distribution called type I Half Logistic Topp-Leone Inverse Weibull distribution with four parameters was derived. The nature of the new distribution with the help of its mathematical and statistical properties such as quantile function, ordinary moments, moment generating function and reliability were studied. The probability density function of the minimum and maximum order statistics for the new distribution was also obtained. A classical estimation of the unknown parameters of the model was done using the technique of maximum likelihood estimation. Monte Carlo simulation study was carried out to see the performance of maximum likelihood estimation method. The proposed model was applied to two real datasets and the results showed that the new model provides a better fit than the competing distributions considered.

Keywords: Monte Carlo simulation, Type I Half Logistic Topp-Leone Inverse Weibull distribution

1 Introduction

Continuous probability distributions have witness great extension or generalizations lately which enable development of wide range of distributions. Thus standard distributions are improved and extended, providing more versatility for applications in a variety of fields. Numerous studies have been conducted on the recently developed distributions, proving their increased adaptability and wide range of applications. Inverse Weibull (IW) distribution is a modified version of the Weibull distribution with altered variables. Its simplicity and adaptability make it appealing for modeling a range of failure characteristics. The Inverse Weibull distribution was first created by Keller and Kanath (1982) to analyze the deterioration of mechanical components in survival and reliability investigations. Some of the generalizations and extensions of the inverse Weibull distribution are the Topp-Leone inverse Weibull distribution by Abbas et al (2017), Odd Frechet Inverse Weibull by Fayomi (2019), (2020), the Extended Inverse Weibull by Alkarni *et al.*, (2020), and the modified Burr XII Inverse Weibull by Bhatti *et al.,* (2020), Topp Leone exponentiated inverse Weibull by Lawal et al (2020), a modified inverse Weibull by Gauthami et al (2023), Adepoju et al (2024), introduced a cosine Marshall-Olkin family of distribution by adding extra shape parameter to modify Weibull distribution. The extension is not limited to IW only other related extension could be seen lately by Isa et al., (2022), Adepoju et al., (2023), Isa et al., (2023), to mention but few. Inference on parameters is another essential factor in modeling life data. Maximum likelihood is the most used why other methods are also adopted to estimate the model parameters. Several authors have developed and adopted, such can be seen in Anabike et al., (2023), Adepoju et al., (2024), Adepoju et al., (2024).

Justification for this study is to develop a modified Inverse Weibull distribution with enhanced shapes that showcase consistence performance among its contemporaries in term of flexibility.

2 The Type I Half Logistic Topp-Leone Inverse Weibull (TIHLTLIW) Distribution

Adepoju et al (2023) proposed a family of continuous distribution called type I half logistic-Topp-Leone-G to add flexibility and improve the fit of the standard distributions. The cdf and pdf of the family are given as:

$$F_{TIHLTL-G}(x;\zeta,\theta,\chi)=\frac{1-\left[1-\left[1-\left(1-H(x;\chi)\right)^{2}\right]^{\theta}\right]^{\zeta}}{1+\left[1-\left[1-\left(1-H(x;\chi)\right)^{2}\right]^{\theta}\right]^{\zeta}}, \quad (1)$$

$$f_{TIHLTL-G}(x;\zeta,\theta,\chi)=\frac{4\zeta\theta h(x;\chi)[1-H(x;\chi)]\left[1-\left(1-H(x;\chi)\right)^{2}\right]^{\theta-1}\left[1-\left[1-\left(1-H(x;\chi)\right)^{2}\right]^{\theta}\right]^{\zeta-1}}{\left[1+\left[1-\left[1-\left(1-H(x;\chi)\right)^{2}\right]^{\theta}\right]^{\zeta}\right]^{2}}, \quad (2)$$

Where $\zeta,\theta>0$ are the shape parameters and χ is a vector of parent distribution parameters

The cdf and pdf of the Inverse Weibull distribution are given as follows:

$$H(x;\delta,\beta)=e^{-\delta x^{-\beta}}, x>0, \delta,\beta>0, \quad (3)$$

$$h(x;\delta,\beta)=\delta\beta e^{-\beta-1}e^{-\delta x^{-\beta}}, \quad (4)$$

Then, to obtain the cdf of TIHLTLIW distribution, equation (3) is inserted into equation (1) as

$$F_{TIHLTLIW}(x;\zeta,\theta,\chi)=\frac{1-\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta}\right]^{\zeta}}{1+\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta}\right]^{\zeta}}, \quad (5)$$

$$f_{TIHLTLIW}(x;\zeta,\theta,\chi)=\frac{4\zeta\theta\delta\beta x^{-\beta-1}e^{-\delta x^{-\beta}}\left[1-e^{-\delta x^{-\beta}}\right]\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta-1}\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta}\right]^{\zeta-1}}{\left[1+\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta}\right]^{\zeta}\right]^{2}}, \quad (6)$$

where $x>0,\zeta,\theta,\delta,\beta>0,$

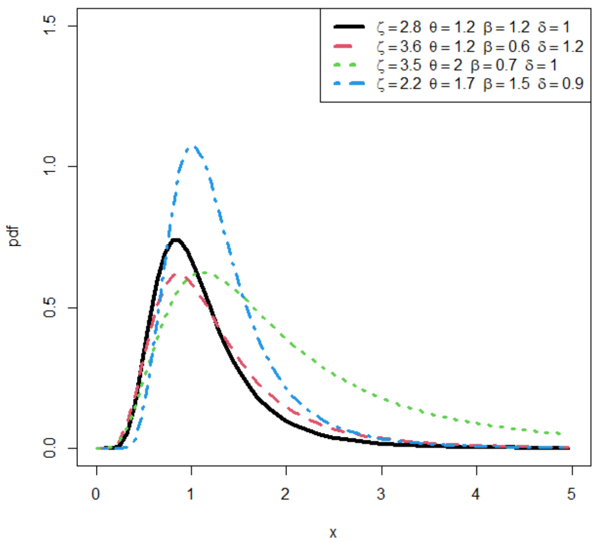

Figure 1. the pdf plot of the TIHLTLIW distribution with distinct values

The figure 1 above depicts the pdf plot of the TIHLTLIW distribution revealing the various shapes the distribution possesses. It is obvious that the distribution is a right skewed.

3 Expansion of Density

In this section, the pdf and cdf of the TIHLTLIW model are expanded using binomial expansion to derive an explicit expression for the model which will be used in the derivation of some of the model's properties.

$$(1-x)^{p}=\sum_{j=0}^{\infty}(-1)^{j}\binom{p}{j}[x]^{j}$$

$$(1+x)^{-p}=\sum_{i=0}^{\infty}(-1)^{i}\binom{p+i-1}{i}[x]^{i} \tag{7}$$

Equation (6) can also be written as

$$f_{TIHLTLIW}(x;\zeta,\theta,\delta,\beta)=4\zeta\theta\delta\beta x^{-\beta-1}e^{-\delta x^{-\beta}}\left[1-e^{-\delta x^{-\beta}}\right]\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta-1}\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta}\right]^{\zeta-1}\left[1+\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta}\right]^{\zeta}\right]^{-2} \tag{8}$$

Now, using equation (7) on the last term in equation (8), we have

$$\left[1+\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta}\right]^{\zeta}\right]^{-2}=\sum_{i=0}^{\infty}(-1)^{j}\binom{1+i}{i}\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta}\right]^{\zeta i}$$

$$\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta}\right]^{\zeta(i+1)-1}=\sum_{i=0}^{\infty}(-1)^{j}\binom{\zeta(i+1)-1}{j}\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta j}$$

$$\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^{2}\right]^{\theta(j+1)-1}=\sum_{k=0}^{\infty}(-1)^{k}\binom{\theta(j+1)-1}{k}\left(1-e^{-\delta x^{-\beta}}\right)^{2k}$$

$$\left[1-e^{-\delta x^{-\beta}}\right]^{2k+1}=\sum_{m=0}^{\infty}(-1)^{m}\binom{2k+1}{m}\left[e^{-\delta x^{-\beta}}\right]^{m}$$

$$f_{TIHLTLIW}(x;\zeta,\theta,\chi) =$$

$$\sum_{i,j,k,m=0}^{\infty}(-1)^{i+j+k+m}\binom{1+i}{i}4\zeta\theta\delta\beta\binom{\zeta(i+1)-1}{j}\binom{\theta(j+1)-1}{k}\binom{2k+1}{m}x^{-\beta-1}\left[e^{-\delta x^{-\beta}}\right]^{m+1} \quad (9)$$

in the same vain, the cdf is also expressed as

$$[F_{TIHLTLIW}(x;\zeta,\theta,\delta,\beta)]^h$$

$$= \left[1-\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^\theta\right]^\zeta\right]^h\left[1+\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^\theta\right]^\zeta\right]^{-h}$$

$$A = \left[1-\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^\theta\right]^\zeta\right]^h = \sum_{w=0}^{h}(-1)^w\binom{h}{w}\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^\theta\right]^{\zeta w}$$

Now consider $\left[1+\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^\theta\right]^\zeta\right]^{-h}$ for expansion, by using the binomial expansion given in equation (7).

Then,

$$B = \left[1+\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^\theta\right]^\zeta\right]^{-h}$$

$$= \sum_{p=0}^{h}(-1)^p\binom{h+p-1}{p}\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^\theta\right]^{\zeta p}$$

Combining $\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^{\theta}\right]^{\zeta w}$ and $\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^{\theta}\right]^{\zeta p}$,

we obtain $\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^{\theta}\right]^{\zeta (w+p)}$

Then,

$$[F_{TIHLTLIW}(x;\zeta,\theta,\delta,\beta)]^h = \sum_{p,w=0}^{h}(-1)^{p+w}\binom{h}{w}\binom{h+p-1}{p}\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^{\theta}\right]^{\zeta (w+p)}$$

$$\left[1-\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^{\theta}\right]^{\zeta (w+p)} = \sum_{q=0}^{\infty}(-1)^q\binom{\zeta(p+w)}{q}\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^{\theta q}$$

$$\left[1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right]^{\theta q} = \sum_{t=0}^{\infty}(-1)^t\binom{q\theta}{t}\left(1-e^{-\delta x^{-\beta}}\right)^{2t}$$

$$\left(1-e^{-\delta x^{-\beta}}\right)^{2t} = \sum_{d=0}^{\infty}(-1)^d\binom{2t}{d}\left(e^{-\delta x^{-\beta}}\right)^{d}$$

$$[F_{TIHLTLIW}(x;\zeta,\theta,\delta,\beta)]^h = \sum_{q,t,d=0}^{\infty}\sum_{p,w=0}^{h}(-1)^{p+w+q+t+d}\binom{h}{w}\binom{h+p-1}{p}\binom{\zeta(p+w)}{q}\binom{\theta q}{t}\binom{2t}{d}\left(e^{-\delta x^{-\beta}}\right)^{d} \qquad (10)$$

4 Statistical Properties

4.1 Probability Weighted Moments

$$\tau_{r,s} = E[X^r F(X)^s] = \int_0^1 x^r f(x)\big(F(x)\big)^s dx$$

The Probability Weighted Moments (PWMs) of the TIHLTLIW distribution are derived by substituting expansion of density and also replacing h with s in equation (10) as

$$\tau_{r,s} = \int_0^1 4\zeta\theta\delta\beta x^r \sum_{i,j,k,m=0}^{\infty}\sum_{q,t,d=0}^{\infty}\sum_{p,w=0}^{h}(-1)^{p+w+q+t+d+i+j+k+m} \binom{\zeta(i+1)-1}{j}$$

$$\binom{\theta(j+1)-1}{k}\binom{2k+1}{m}\binom{h}{w}\binom{h+p-1}{p}\binom{\zeta(p+w)}{q}\binom{\theta q}{t}\binom{2t}{d}\binom{1+i}{i} x^{-\beta-1}\left[e^{-\delta x^{-\beta}}\right]^{m+d+1} dx \quad (11)$$

Consider the integral part in equation (11)

$$\int_0^{\infty} x^{r-\beta-1}\left(e^{-\delta x^{-\beta}}\right)^{d+m+1} dx$$

Let $y = (d+m+1)\delta x^{-\beta}, \Longrightarrow x = \left[\frac{y}{(d+m+1)\delta}\right]^{\frac{-1}{\beta}}; dx = \frac{dy}{(d+m+1)\delta x^{-\beta-1}}$

$$\psi_I = 4\zeta\theta\left[\frac{1}{(d+m+1)\delta}\right]^{\frac{-r-\beta}{\beta}} \delta^{\frac{-r-\beta}{\beta}} \sum_{i,j,k,m=0}^{\infty}\sum_{q,t,d=0}^{\infty}\sum_{p,w=0}^{h}(-1)^{p+w+q+t+d+i+j+k+m}$$

$$\binom{\zeta(i+1)-1}{j}\binom{\theta(j+1)-1}{k}\binom{2k+1}{m}\binom{h}{w}\binom{h+p-1}{p}\binom{\zeta(p+w)}{q}\binom{\theta q}{t}\binom{2t}{d}\binom{1+i}{i}$$

Then, its obtained as $\tau_{r,s} = \psi_I \Gamma\left(1 - \frac{r-\beta-1}{\beta}\right)$ (12)

4.2 Moments

$$E(X^r) = \int_0^\infty x^r f(x)dx \tag{13}$$

$$E(X^r) = \int_0^\infty 4\zeta\theta\delta\beta \sum_{i,j,k,m=0}^\infty (-1)^{i+j+k+m} \binom{1+i}{i} \binom{\zeta(i+1)-1}{j} \binom{\theta(j+1)-1}{k} \binom{2k+1}{m} x^{r-\beta-1} \left[e^{-\delta x^{-\beta}}\right]^{m+1} \tag{14}$$

Where $\nabla_I=$

$$4\zeta\theta \left[\frac{1}{(d+m+1)}\right]^{\frac{-r-\beta}{\beta}} \delta^{\frac{-r-\beta}{\beta}} \sum_{i,j,k,m=0}^\infty (-1)^{i+j+k+m} \binom{1+i}{i} \binom{\zeta(i+1)-1}{j} \binom{\theta(j+1)-1}{k} \binom{2k+1}{m}$$

By integrating the integral part in equation (14)

Similarly, the moment is obtained as

$$E(X^r) = \nabla_I \Gamma\left(1 - \frac{r-\beta-1}{\beta}\right) \tag{15}$$

4.3 Mean

The mean of TIHLTLIW distribution is obtained by setting $r = 1$ in equation (15) as

$$\Delta_I^1 = 4\zeta\theta \left[\frac{1}{(d+m+1)}\right]^{\frac{-1-\beta}{\beta}} \delta^{\frac{-1-\beta}{\beta}} \sum_{i,j,k,m=0}^\infty (-1)^{i+j+k+m} \binom{1+i}{i} \binom{\zeta(i+1)-1}{j} \binom{\theta(j+1)-1}{k} \binom{2k+1}{m}$$

$$E(X) = \Delta_I^1 \Gamma(2) \tag{16}$$

4.4 Moment generating function (mgf)

The mgf is given as:

$$M_x(t) = \int_0^\infty e^{tx} f(x)dx \quad (17)$$

The mgf of TIHLTLIW distribution is obtained as

$$M_x(t) = 4\zeta\theta\delta\beta \sum_{i,j,k,m=0}^{\infty} (-1)^{i+j+k+m} \binom{1+i}{i} \binom{\zeta(i+1)-1}{j} \binom{\theta(j+1)-1}{k} \binom{2k+1}{m} \int_0^\infty e^{tx} \left[e^{-\delta x^{-\beta}}\right]^{m+1} x^{-\beta-1} dx$$

By expanding $e^{tx} = \sum_{b=0}^{\infty} \frac{t^b x^b}{b!}$ and following the process for deriving moments as outlined above,

$$M_x(t) = \int_0^\infty 4\zeta\theta \left[\frac{1}{(d+m+1)}\right]^{\frac{-1-\beta}{\beta}} \delta^{\frac{-1-\beta}{\beta}} \sum_{i,j,k,m=0}^{\infty} (-1)^{i+j+k+m} \binom{1+i}{i} \binom{\zeta(i+1)-1}{j} \binom{\theta(j+1)-1}{k} \binom{2k+1}{m} \Gamma\left(1-\frac{b}{\beta}\right) \quad (18)$$

4.5 Reliability Function

$$R(x;\zeta,\theta,\delta,\beta) = \frac{2\left[1-\left(1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right)^\theta\right]^\zeta}{1+\left[1-\left(1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right)^\theta\right]^\zeta} \quad (19)$$

4.6 Hazard Function

$$T(x;\zeta,\theta,\delta,\beta) \frac{4\zeta\theta\delta\beta x^{-\beta-1} e^{-\delta x^{-\beta}}\left(1-e^{-\delta x^{-\beta}}\right)\left(1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right)^{\theta-1}\left[1-\left(1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right)^\theta\right]^\zeta}{1+\left[1-\left(1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right)^\theta\right]^\zeta 2\left[1-\left(1-\left(1-e^{-\delta x^{-\beta}}\right)^2\right)^\theta\right]^\zeta} \quad (20)$$

4.7 Quantile Function

The quantile function of the TIHLTLIW distribution is expressed below

$F_{TIHLTLIW}(x;\zeta,\theta,\delta,\beta)=u$

Then, obtained as

$$x=Q(u)=\left[\frac{-1}{\delta}log\left(1-\left(1-\left(1-\left(\frac{1-u}{u+1}\right)^{\frac{1}{\zeta}}\right)^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right)\right]^{\frac{-1}{\beta}} \quad (21)$$

5 Order Statistics

$$f_{r:n}(x;\zeta,\theta,\delta,\beta)=\frac{f(x)}{B(r,n-r-1)}\sum_{v=0}^{n-r}(-1)^{v}\binom{n-r}{v}[F(x)]^{v+r-1} \quad (22)$$

The pdf of the r^{th} order statistic is derived by substituting equations (9) and (10). Additionally, by replacing h with $v+r-1$ in equation (10), we obtain the following expression.

$$f_{r:n}(x;\zeta,\theta,\delta,\beta)$$
$$=\frac{4\zeta\theta\delta\beta x^{-\beta-1}}{B(r,n-r+1)}\sum_{v=0}^{n-r}\sum_{i,j,k,m=0}^{\infty}\sum_{q,t,d=0}^{\infty}\sum_{p,w}^{v+r-1}(-1)^{p+w+q+t+d}$$
$$\times\binom{1+i}{i}\binom{\zeta(1+i)-1}{j}\binom{\theta(1+j)-1}{k}\binom{2k+1}{m}\binom{v+r-1}{w}\binom{v+r+p-2}{p}\binom{\zeta(p+w)}{q}$$
$$\times\binom{\theta q}{t}\binom{2t}{d}\binom{n-r}{v}\left(e^{-\delta x^{-\beta}}\right)^{m+1+d+v+r-1} \quad (23)$$

The pdf of the minimum order statistic is obtained by setting $r=1$ in equation (23), resulting in the following expression.

$f_{1:n}(x;\zeta,\theta,\delta,\beta)$

$$= \frac{4\zeta\theta\delta\beta x^{-\beta-1}}{B(1,n)} \sum_{v=0}^{n-1} \sum_{i,j,k,m=0}^{\infty} \sum_{q,t,d=0}^{\infty} \sum_{p,w}^{v} (-1)^{p+w+q+t+d}$$

$$\times \binom{1+i}{i} \binom{\zeta(1+i)-1}{j} \binom{\theta(1+j)-1}{k} \binom{2k+1}{m} \binom{v}{w} \binom{v+p-1}{p} \binom{\zeta(p+w)}{q}$$

$$\times \binom{\theta q}{t} \binom{2t}{d} \binom{n-1}{v} \left(e^{-\delta x^{-\beta}}\right)^{m+1+d+v} \qquad (24)$$

Similarly, the pdf of the maximum order statistic is derived by setting $r = n$ in equation (23), resulting in the following expression.

$f_{n:n}(x;\zeta,\theta,\delta,\beta)$

$$= \frac{4\zeta\theta\delta\beta x^{-\beta-1}}{B(n,1)} \sum_{v=0}^{n-n} \sum_{i,j,k,m=0}^{\infty} \sum_{q,t,d=0}^{\infty} \sum_{p,w}^{v+n-1} (-1)^{p+w+q+t+d}$$

$$\times \binom{1+i}{i} \binom{\zeta(1+i)-1}{j} \binom{\theta(1+j)-1}{k} \binom{2k+1}{m} \binom{v+n-1}{w} \binom{v+n+p-2}{p} \binom{\zeta(p+w)}{q}$$

$$\times \binom{\theta q}{t} \binom{2t}{d} \binom{1}{v} \left(e^{-\delta x^{-\beta}}\right)^{m+1+d+v+n-1} \qquad (25)$$

6 Maximum Likelihood Estimation

Let $x_1, x_2, \ldots x_n$ be random sample of size n from the TIHLTLIW$(\zeta,\theta,\delta,\beta)$ distribution. Then the sample log-likelihood function of the TIHLTLIW$(\zeta,\theta,\delta,\beta)$ distribution is obtained as

$log(L) = nlog(4) + nlog(\zeta) + nlog(\theta) + nlog(\delta) + nlog(\beta) - (\beta-1)\sum_{i=1}^{n} log(x_i) +$

$(\zeta-1)\sum_{i=1}^{n} log\left[1-\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^{\theta}\right] - 2\sum_{i=1}^{n} log\left[1+\left[1-\left(1-\left(1-\right.\right.\right.\right.$

$$e^{-\delta x_i^{-\beta}}\Big)^2\Big)^\theta\Big]^\zeta\Big] - \delta\sum_{i=1}^n x_i^{-\beta} + \sum_{i=1}^n log\left[1 - e^{-\delta x_i^{-\beta}}\right] + (\theta-1)\sum_{i=1}^n\left[1 - \left(1 - e^{-\delta x_i^{-\beta}}\right)^2\right] \tag{26}$$

Differentiating the log-likelihood function with respect to $\zeta, \theta, \delta, \beta$ and setting the result to zero, we obtain:

$$\frac{\partial L}{\partial \zeta} = \frac{n}{\zeta} + \sum_{i=1}^n log\left[1 - \left(1 - \left(1 - e^{-\delta x_i^{-\beta}}\right)^2\right)^\theta\right]$$

$$-2\sum_{i=1}^n \frac{\left[1-\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^\theta\right]^\zeta log\left[1-\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^\theta\right]}{\left[1+\left[1-\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^\theta\right]^\zeta\right]} \tag{27}$$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^n log\left(1 - \left(1 - e^{-\delta x_i^{-\beta}}\right)^2\right) - (\zeta-1)\sum_{i=1}^n \frac{\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^\theta log\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)}{\left[1-\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^\theta\right]}$$

$$+2\sum_{i=1}^n \frac{\zeta\left(1-\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^\theta\right)^{\zeta-1}\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^\theta log\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)}{\left[1+\left[1-\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^\theta\right]^\zeta\right]} \tag{28}$$

$$\frac{\partial L}{\partial \delta} = \frac{n}{\delta} + \sum_{i=1}^n x_i^{-\beta} - \sum_{i=1}^n \frac{e^{-\delta x_i^{-\beta}} x_i^{-\beta}}{\left[1-e^{-\delta x_i^{-\beta}}\right]} + [\theta-1]\sum_{i=1}^n \frac{2\left[1-e^{-\delta x_i^{-\beta}}\right]e^{-\delta x_i^{-\beta}} x_i^{-\beta}}{\left[1-\left[1-e^{-\delta x_i^{-\beta}}\right]^2\right]}$$

$$+2\sum_{i=1}^n \frac{\zeta\left(1-\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^\theta\right)^{\zeta-1}\theta\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^{\theta-1}2\left(1-e^{-\delta x_i^{-\beta}}\right)e^{-\delta x_i^{-\beta}} x_i^{-\beta}}{\left[1+\left(1-\left(1-\left(1-e^{-\delta x_i^{-\beta}}\right)^2\right)^\theta\right)^\zeta\right]}$$

$$(\zeta-1)\sum_{i=1}^{n}\frac{\theta\left(1-\left(1-e^{-\delta x_i{}^{-\beta}}\right)^2\right)^{\theta-1}2\left(1-e^{-\delta x_i{}^{-\beta}}\right)e^{-\delta x_i{}^{-\beta}}x_i{}^{-\beta}}{\left[1-\left(1-\left(1-e^{-\delta x_i{}^{-\beta}}\right)^2\right)^{\theta}\right]} \quad (29)$$

$$\frac{\partial L}{\partial \beta}=\frac{n}{\beta}-\sum_{i=1}^{n}log(x_i)+$$

$$\delta\sum_{i=1}^{n}x_i^{-\beta}log\,x_i-\sum_{i=1}^{n}\frac{e^{-\delta x_i^{-\beta}}\delta x_i^{-\beta}logx_i}{\left[1-e^{-\delta x_i^{-\beta}}\right]}+$$

$$[\theta-1]\sum_{i=1}^{n}\frac{2\left[1-e^{-\delta x_i^{-\beta}}\right]e^{-\delta x_i^{-\beta}}\delta x_i^{-\beta}logx_i}{\left[1-\left[1-e^{-\delta x_i^{-\beta}}\right]^2\right]}$$

$$+2\sum_{i=1}^{n}\frac{\zeta\left(1-\left(1-\left(1-e^{-\delta x_i{}^{-\beta}}\right)^2\right)^{\theta}\right)^{\zeta-1}\theta\left(1-\left(1-e^{-\delta x_i{}^{-\beta}}\right)^2\right)^{\theta-1}2\left(1-e^{-\delta x_i{}^{-\beta}}\right)e^{-\delta x_i{}^{-\beta}}x_i{}^{-\beta}logx_i}{\left[1+\left(1-\left(1-\left(1-e^{-\delta x_i{}^{-\beta}}\right)^2\right)^{\theta}\right)^{\zeta}\right]}$$

$$(\zeta-1)\sum_{i=1}^{n}\frac{\theta\left(1-\left(1-e^{-\delta x_i{}^{-\beta}}\right)^2\right)^{\theta-1}2\left(1-e^{-\delta x_i{}^{-\beta}}\right)e^{-\delta x_i{}^{-\beta}}x_i{}^{-\beta}logx_i}{\left[1-\left(1-\left(1-e^{-\delta x_i{}^{-\beta}}\right)^2\right)^{\theta}\right]} \quad (30)$$

These equations [(27), (28), (29), (30)] are nonlinear and cannot be solved analytically. Therefore, statistical software such as *R* with iterative numerical techniques is required to obtain the value of the unknown parameters.

7 Simulation study

This section addresses a numerical analysis to evaluate the performance of MLE for TIHLTLIW distribution.

Table 1: MLEs, biases and RMSE for some values of the parameters of TIHLTLIW distribution

		(1.5,2.5,3,3.1)			(2,2.9,2.8,2.5)		
N	Parameters	Estimated Values	Bias	RMSE	Estimated Values	Bias	RMSE
	ζ	1.7451	0.2451	1.0109	2.1720	0.1720	1.0148
20	δ	2.9423	0.4423	1.0308	3.3573	0.4573	1.2316
	θ	3.1601	0.1601	0.9145	3.0430	0.2430	0.9227
	β	3.5092	0.4092	1.1262	2.8416	0.3416	0.9293
	ζ	1.5983	0.0983	0.6619	2.1051	0.1051	0.7437
50	δ	2.7068	0.2068	0.6087	3.0652	0.1652	0.6144
	θ	3.0236	0.0236	0.5835	2.8860	0.0860	0.5694
	β	3.3202	0.2202	0.7644	2.6374	0.1374	0.5383
100	ζ	1.5440	0.0440	0.4923	2.0607	0.0607	0.5694
	δ	2.6106	0.1106	0.3849	2.9654	0.0654	0.3793
	θ	3.0100	0.0100	0.4519	2.8694	0.0694	0.4263
	β	3.2301	0.1301	0.5267	2.5681	0.0681	0.3496
250	ζ	1.5258	0.0258	0.2921	2.0356	0.0356	0.3550
	δ	2.5252	0.0252	0.2110	2.9124	0.0124	0.2137
	θ	3.0340	0.0340	0.3649	2.8396	0.0396	0.3101
	β	3.1328	0.0328	0.3075	2.5157	0.0157	0.2118
500	ζ	1.5129	0.0129	0.1894	2.0089	0.0089	0.2285
	δ	2.5119	0.0119	0.1413	2.9069	0.0069	0.1304
	θ	3.0079	0.0079	0.2515	2.8123	0.0123	0.1935
	β	3.1099	0.0099	0.1914	2.5094	0.0094	0.1334
1000	ζ	1.5071	0.0071	0.1155	2.0028	0.0028	0.1324
	δ	2.5005	0.0005	0.0983	2.8968	-0.0032	0.0808
	θ	3.0078	0.0078	0.1659	2.8087	0.0087	0.1160
	β	3.1019	0.0019	0.1180	2.5028	0.0028	0.0748

Table 1 displays the values of biases, estimated values and RMSEs. It is noticed from the table that the RMSEs approach zero and the estimates tend to the true parameter values as the sample size increases. This is an indication that the maximum likelihood estimates are efficient and consistent estimator of the TIHLTLIW distribution's parameters.

8 Applications

The fit of TIHLTLIW distribution is tested with applications to real-life data sets to assess its flexibility and robustness. The fit of the TIHLTLIW model is compared with some existing distributions having inverse weibull distribution as their baseline. The comparators are:

- The TIHLIW distribution developed by Alkarni *et al.*, (2020).
- The Marshall-Olkin extended inverse Weibull (MOIW) distribution developed by Pakungwati *et al.*, (2018).
- The generalized inverse Weibull (GIW) distribution proposed by De Gusmao *et al.*, (2011).
- The Kumaraswamy–Inverse Weibull (KIW) distribution proposed by Shahbaz *et al.*, (2012).
- The IW distribution developed by Keller and Kanath (1982).

The first data set shown below represents the remissions times (in months) of a random sample of one hundred and twenty-eight (128) bladder cancer patients, previously used by Bello *et al.*, *(*2023)

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02,4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25,8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

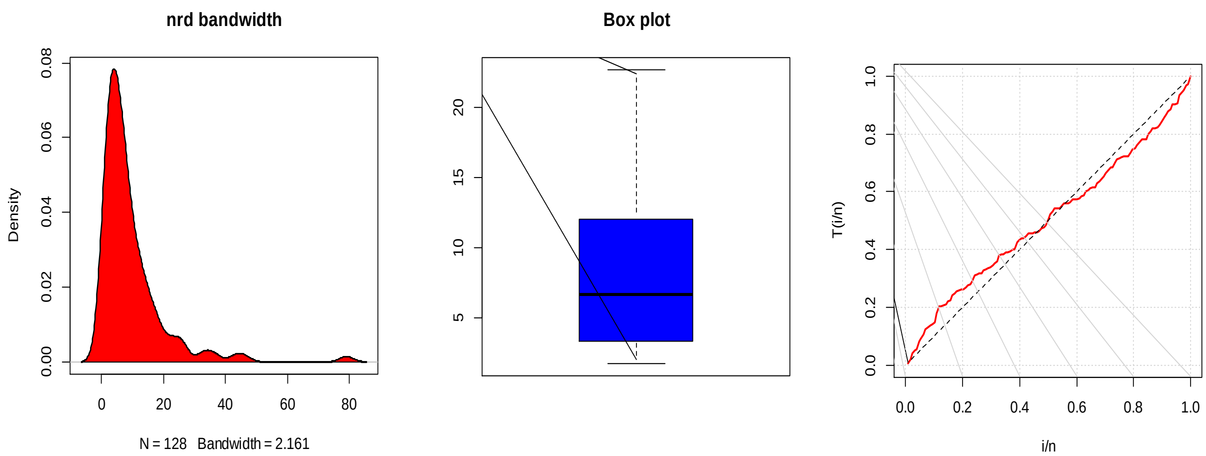

Figure 2: the kernel density, the boxplot and TTT plot of the data set 1.

Table 2: The models' MLEs and performance requirements based on data set 1

Distributions	δ	ζ	θ	β	LL	AIC
TIHLTLIW	**0.7295**	**0.2012**	**0.2818**	**1.5138**	**-16.2535**	**40.50693**
TIHLIW	4.1888	8.1994	-	0.4021	-41.7416	89.4832
KIW	3.7629	12.5436	1.4278	0.3518	-31.6601	71.3202
GIW	31.3253	1.9671		0.7521	-44.0008	94.0016
MOIW	29.0212	-	2.4573	1.3083	-42.2386	90.4772
IW	2.4309	0.7519	-	-	-45.4008	94.8016

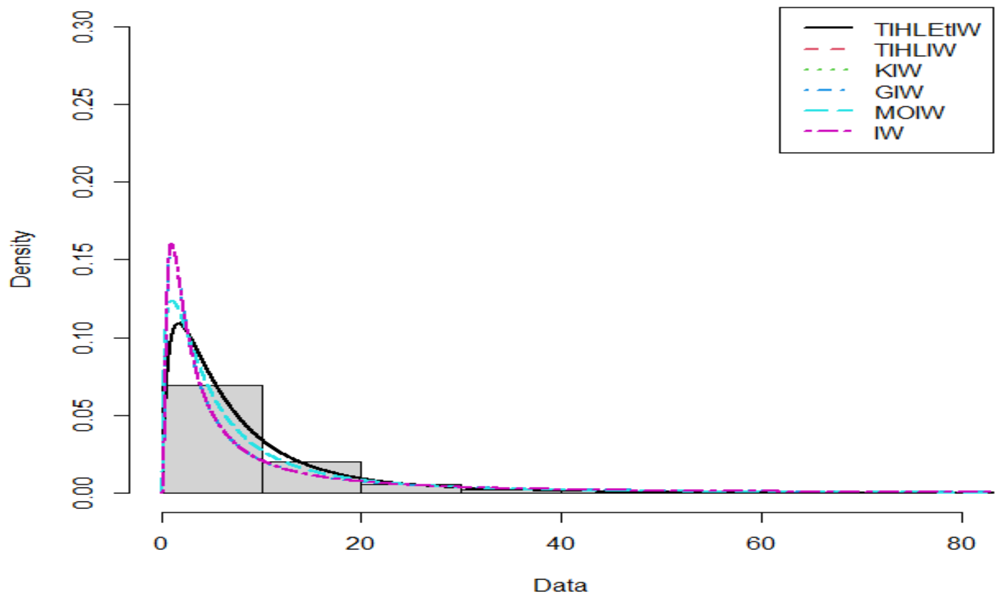

Figure 3: Histogram of the fitted distributions for data set 1.

The figure 3 revealed the fitness of the TIHLTLIW and its competitors. It is obvious that the TIHLTLIW model fitted the data set 1 better than the competing models

The second data set shown below represents the survival times (in days) of seventy two (72) guinea pigs infected with virulent tubercle bacilli. It has been previously used by Bello et al., (2021)

12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83,84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

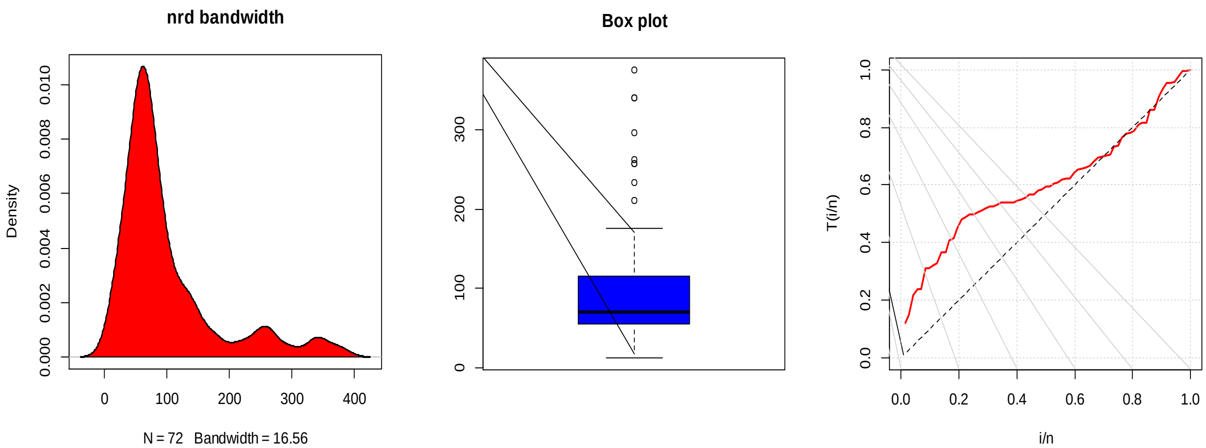

Figure 4: the kernel density, the boxplot and TTT plot of the data set 2.

Table 3: The models' MLEs and performance requirements based on data set 2

Distributions	δ	ζ	θ	β	LL	AIC
TIHLTLIW	7.7011	3.1376	4.9432	0.5781	-235.153	478.306
TIHLIW	18.4850	9.8337	-	0.4948	-392.7921	791.5842
KIW	10.5995	3.7510	6.7293	0.6593	-390.3068	788.6136
GIW	5.2434	6.5694		0.8805	-410.47	826.94
MOIW	4.7384	-	0.0342	1.8070	-392.6236	791.2472
IW	4.8332	0.5069	-	-	-447.6698	899.3396

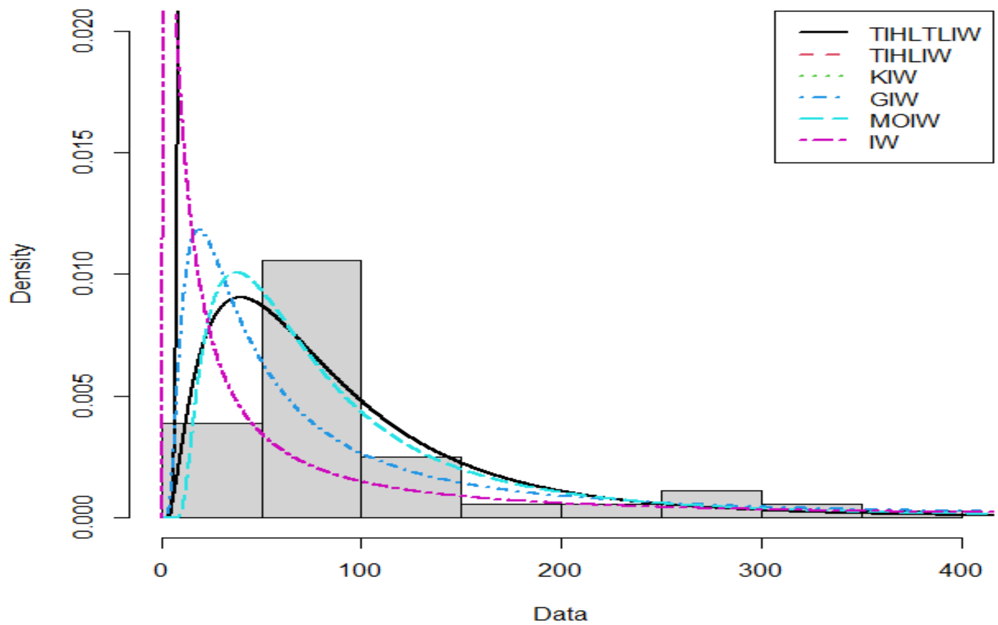

Figure 3: Histogram of the fitted distributions for data set 2.

The figure 3 revealed the fitness of the TIHLTLIW and its competitors. It is obvious that the TIHLTLIW model fitted the data set 2 better than the competing models

The outcomes of the MLE of the TIHLTLIW distribution's parameters, together with the comparator distributions, are shown in Tables 2 and 3. The new model obtained the lowest AIC value according to the goodness of fit metric AIC, indicating that the TIHLTLIW distribution best matches the two data sets.

9 Conclusion

The type I half logistic Topp-Leone inverse Weibull distribution is a novel continuous probability distribution that was proposed and examined in this study. Using the inverse Weibull distribution as the baseline distribution, the new model was developed from the type I half logistic Topp-Leone-G family of distributions. The properties of the new model such as probability-weighted moments, moments, quantile function, moment generating function, reliability function, hazard function, and order statistics were examined as statistical components of the newly proposed model. The parameters of the model are estimated using the method of maximum likelihood technique. The effectiveness of the estimation technique was evaluated by analyzing simulation data to analyze the performance of the new distribution. To determine the significance and adaptability of the novel continuous distribution, two real data sets were used. The findings show that the new model seems to be better than the models that were previously taken into consideration.

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