A NEW BURR TYPE II DISTRIBUTION: PROPERTIES AND APPLICATION

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ABSTRACT

Burr Type II distribution (BIID), a type from the Burr family of distributions is used in survival and reliability analysis. It is also used in modeling skewed data such as in finance and in hydrology. To improve the flexibility of BIID in fitting different data sets, this study derived a new three parameter BIID termed the New Burr Type II Distribution (NBIID). The underlying characteristics of the new distribution were studied. Mean Square Error (MSE) was adopted as a measure for evaluating the efficiency and consistency of the two parameter estimation methods, Maximum Likelihood Estimation (MLE) and Maximum Product of Spacing (MPS) proposed for estimating NBIID parameters. Results indicate MPS a more efficient and consistent parameter estimation method for NBIID. Furthermore, four model selection metrics; Bayesian Information Criterion (BIC), Akaike Information Criterion (AIC), Hannan-Quinn Information Criterion (HQIC) and Consistent Akaike Information Criterion (CAIC) were used to evaluate the performance of NBIID amongst two comparative ones using four real-life data sets. The NBIID performed better, demonstrating that it can provide a better fit in comparison to comparative models and was concluded a good choice for modeling different real-data sets.

Keywords: Burr Type II Distribution; Probability Density Function; Cumulative Density Function; Maximum Likelihood Estimation; Maximum Product of Spacing.

1. Introduction

New distributions and classes of distributions are being introduced and studied as extensions of existing distributions and classes of distributions. Studies have shown that these new distributions and classes are usually more flexible and provide good fit to data when compared with their counterparts. Statistical distributions are useful tools in fields such as health, finance, meteorology, insurance, and many others for comprehensive study of data sets. There are various ways to expand or modify current distributions to create new ones with more flexibility. Among these approaches are the exponentiation technique developed by Gompertz (1825); differential equation method developed by Pearson

(1895). Hasting *et al.* (1947) developed quantile function approaches; Johnson (1949) presented a transformation method; Shaw and Buckley (2009) proposed the transmutation method; Gupta and Kundu (2009) reviewed the Power Transformed Method (PTM). Alzaatreh *et al.* (2013) was credited with the transform transformer strategy; α-power transformation method was introduced by Mahdavi and Kundu (2017). and Al-*Babtain et al.* (2020) presented the Modified Kies generator (MK-G).

There are twelve different kinds of cumulative distribution functions in the Burr system of distributions. The Burr type II distribution (BIID) is one of the twelve types of continuous distributions in the Burr system. The Burr system of distributions is the special case or limiting case of several common theoretical distributions, such as the logistic Weibull, Gompertz, exponential, generalized logistic, normal, extreme value, and uniform distributions.

Gupta and Kundu (2010) introduced the proportional reversed hazard logistic distribution which can be identified as the BIID. The BIID is also known as the type I generalized logistic distribution. The Probability Density Function (PDF) can be skewed and is unimodal in nature and log-concave for all values of the shape parameter, Literature has a variety of Burr distribution adjustments. Yari and Tondpour (2017) derived the new Burr distribution by utilizing a differential equation to which Burr distributions are solutions. Isa *et al.* (2022) introduced Sine Burr XII distribution by trigonometric transformation using Burr XII distribution as the base distribution. The new distribution has no additional parameter and the properties were studied. The MLE method was used for the estimation of distribution parameters and two data sets were used to demonstrate the applicability of the Sine-Burr XII distribution.

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Olapede (2004) extended the BIID obtaining the extended type I generalized logistic distribution. Nahed (2017) derived the transmuted BIID, an extension of the BIID. Alshenawy *et al.* (2022) investigated the posterior estimation for the Burr type II distribution's parameters. For the posterior estimation, several loss functions and informative and noninformative priors were considered. Using actual data, the suggested model fitting abilities was evaluated against seven classes of distributions. The comparison took into account the generalizations of the following distributions: Gompertz, half normal, log-logistic, Laplace, Weibull, exponential, Rayleigh, gamma, log normal, Pareto, Levy, inverse gamma, chi-square, Maxwell, and inverse chi-square distributions. The Kolmogorov-Smirnov (KS) test, BIC, and AIC were used in the comparison. The study's findings indicated that the BIID and their modifications is a suitable substitute for lifetime distributions that are frequently employed. Comparable outcomes were obtained using some standards and modified distributions with up to six parameters in the model.

Bhatti *et al.* (2023) introduced the New Modified Burr XII (NMBXII) distribution which is compounding of gamma distribution and the generalized Nadarajah-Haghighi as well as a derivation from the T-X family modification method. The MLE method was adopted for the estimation of model parameters. Empirically authors established NMBXII suitable for study of time periods between successive earthquakes and flood discharges and tax revenue. Anafo *et al.* (2024) modified the Burr XII distribution developing the modified alpha power-transformed Burr XII distribution by the integration of the weighted version of the alpha power transformation family of distribution. Other adjustments of Burr distribution include: include: New modified Burr type III distribution (Jamal *et al.,* 2021), Mc-Donald modified Burr type III (Mukhtar *et al.,* 2019), Kumaraswamy Burr-Type X (Madaki *et al.,* 2022), Beta Burr XII (Paranaiba *et al.,* 2011), Kumaraswamy Burr XII (Paranaiba *et al.,* 2012), Type II Exponentiated Half Logistic-Odd Burr X-G Power Series

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(Dingalo *et al.*, 2023), Power Unit Burr XII distribution (Yildirim *et al.*, 2023)), Logistic Burr XII (Guerra *et al.*, 2023) and Generalized Gamma Burr III (Olobatuyi *et al.*, 2018). While certain members of Burr family of distributions have been utilized in successful simulation and analysis of lifetime data, other family members are not studied as much. BIID has not been given much consideration in lifetime modelling, even though it is the second example of a solution to the differential equation specified by the Burr system of distribution.

This work focuses on a new variant of the BIID termed the New Burr Type II Distribution (NBIID), building on these earlier studies. To make the distribution more useful in the modeling of data with extreme values or heavy tails whose shapes may not have been sufficiently described by the existing distribution, the NBII distribution introduces two additional parameters by transformation similar to those described by Nadarajah and Haghighi (2011). The generated distribution's Cumulative Density Function (CDF), PDF, Survival Function (SF), Hazard Function (HF), quantile function, median and order statistics are all derived and validity of PDF tested. Furthermore, the parameters are estimated by method of maximum likelihood and maximum product of spacings and performance compared among comparative distributions. Extending existing distributions provide users with more flexible distributions for modeling of different types of data.

2. Methods

2.1 A New Burr Type II Distribution

From the Burr system of distribution (Burr, 1942), Let Y represent the Burr Type II distributed random variable with CDF and PDF given in Equations (2.1) and (2.2) respectively,

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$$F\left(y\right)=\left(1+e^{-y}\right)^{-\alpha}, \quad \alpha>0,\ -\infty<y<\infty \tag{2.1}$$

and

$$f\left(y\right)=\alpha e^{-y}\left(1+e^{-y}\right)^{-(\alpha+1)} \quad \alpha>0,\ -\infty<y<\infty \tag{2.2}$$

where α is a shape parameter of BIID.

Considering the transformation approach similar to Nadarajah and Haghighi (2011), let

$$y=\left(1+x^{\beta}\right)^{\theta}-1 \quad , \quad \beta,\theta>0; \quad x\in R \tag{2.3}$$

where β and θ are two extra shape parameters. And substituting Equation (2.3) into Equation (2.1) gives the new distribution (NBIID) with CDF given as:

$$F\left(x\right)=\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-\alpha}, \quad \alpha,\beta,\theta>0; \quad -\infty<x<\infty \tag{2.4}$$

where α,β,θ are shape parameters of the distribution.

The PDF corresponding to Equation (2.4) is obtained by differentiating Equation (2.4) with respect to x as:

$$f\left(x\right)=\frac{d}{dx}F\left(x\right)$$

$$=-\alpha\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-\alpha-1} e^{1-\left(1+x^{\beta}\right)^{\theta}}\left[-\theta\left(1+x^{\beta}\right)^{\theta-1}\beta x^{\beta-1}\right]$$

$$f\left(x\right)=\alpha\beta\theta x^{\beta-1}\left(1+x^{\beta}\right)^{\theta-1}e^{1-\left(1+x^{\beta}\right)^{\theta}}\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-\alpha-1} \tag{2.5}$$

The PDF and CDF plot are given in Figure 2.1 and Figure 2.2 respectively.

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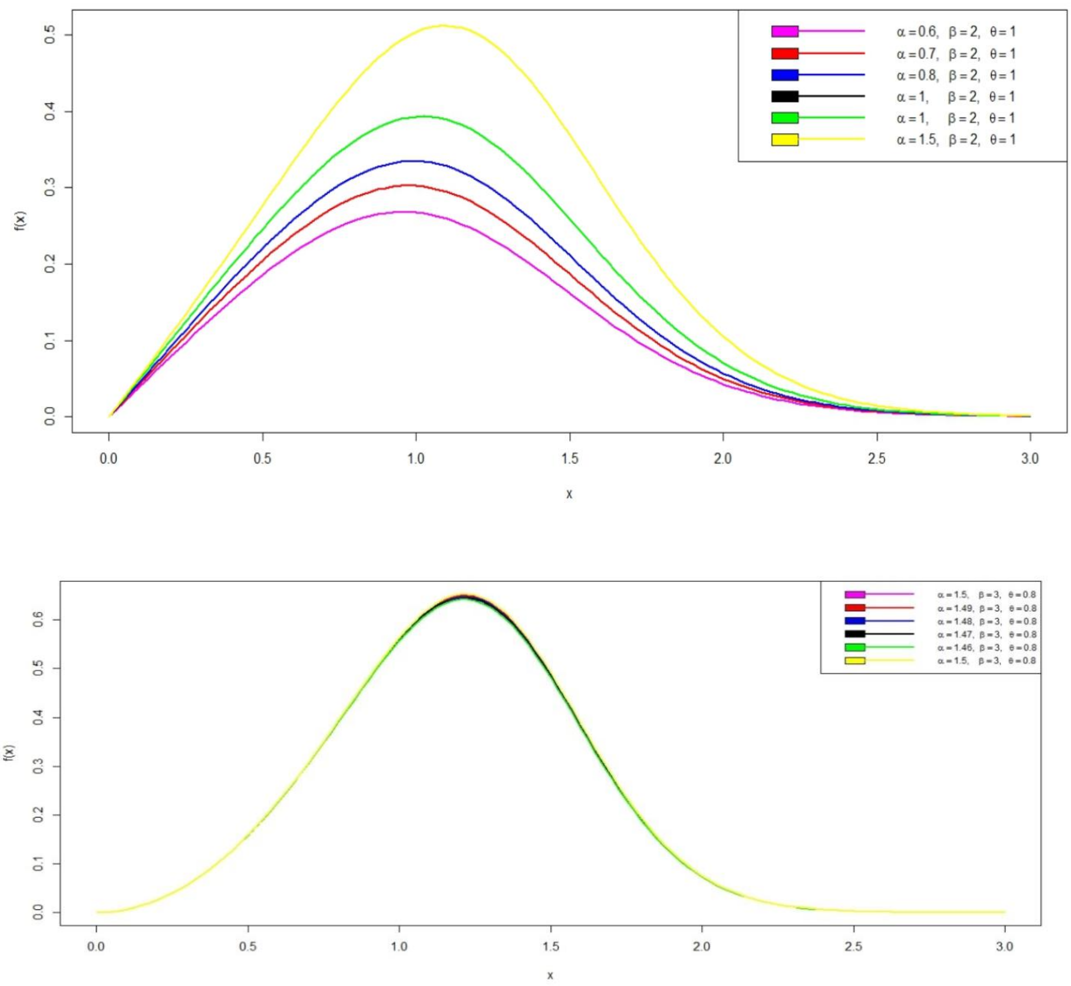
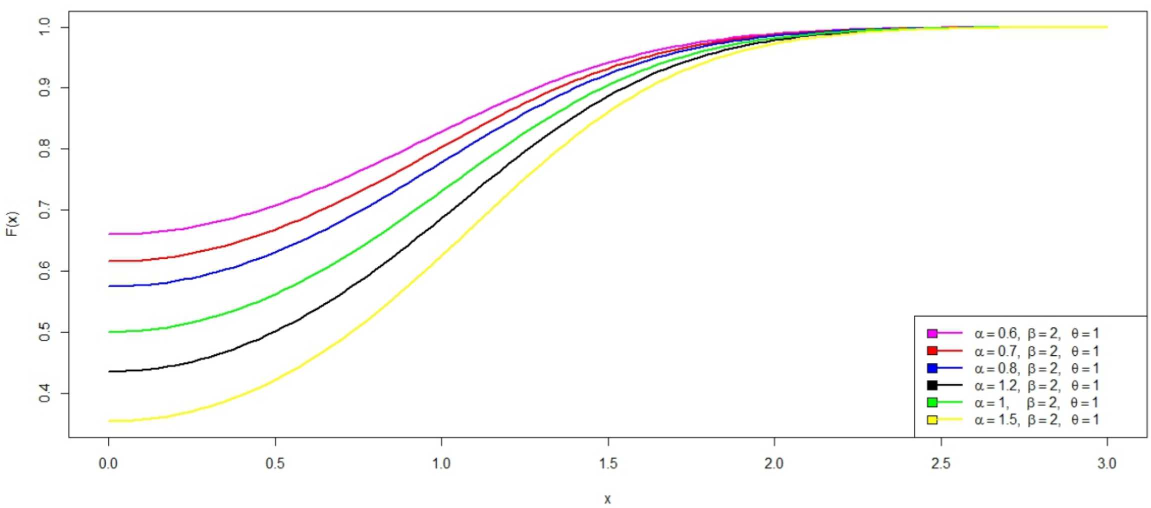


Figure 2. 1: The PDF Plots of NBIID



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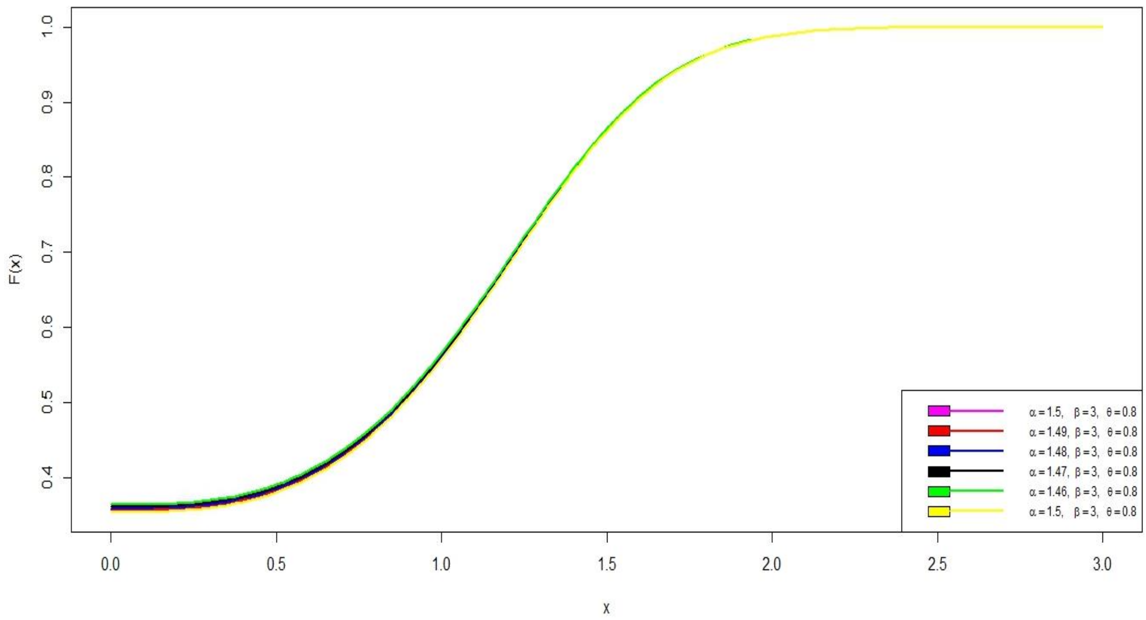

Figure 2. 2: The CDF Plot of NBIID

2.2 Validity Test of New Burr Type II Distribution

It is necessary to show that $\int_{-\infty}^{\infty} f(x)\,dx = 1$. For the proposed distribution,

$$\int_{-\infty}^{\infty} f(x)\,dx = \alpha\beta\theta\int_{-\infty}^{\infty} x^{\beta-1}\left(1+x^{\beta}\right)^{\theta-1} e^{1-\left(1+x^{\beta}\right)^{\theta}}\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-\alpha-1} dx \tag{2.6}$$

Let $m = e^{1-\left(1+x^{\beta}\right)^{\theta}}$

$$dm = e^{1-\left(1+x^{\beta}\right)^{\theta}} - \theta\left(1+x^{\beta}\right)^{\theta-1}\beta x^{\beta-1}dx$$

$$\Rightarrow dx = \frac{-dm}{\theta\beta x^{\beta-1}\left(1+x^{\beta}\right)^{\theta-1} e^{1-\left(1+x^{\beta}\right)^{\theta}}} \tag{2.7}$$

As $x \to \infty;\ m \to \infty$ and as $x \to -\infty;\ m \to 0$

Putting Equation (2.7) into Equation (2.6)

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$$\int_{-\infty}^{\infty} f(x)\,dx = -\alpha\, \cancel{\beta\theta} \int_{\infty}^{0} \cancel{x^{\beta-1}} \left(\cancel{1+x^{\beta}}\right)^{\theta-1} \cancel{e^{1-\left(1+x^{\beta}\right)^{\theta}}} \left(1+m\right)^{-\alpha-1} . \frac{dm}{\cancel{\theta\beta}\, \cancel{x^{\beta-1}} \left(\cancel{1+x^{\beta}}\right)^{\theta-1} \cancel{e^{1-\left(1+x^{\beta}\right)^{\theta}}}}$$

$$= \alpha \int_{0}^{\infty} \left(1+m\right)^{-\alpha-1} dm \qquad (2.8)$$

Let $\frac{1}{A} = 1+m \Rightarrow A = \frac{1}{1+m} \Rightarrow dm = -\left(1+m\right)^{2} dA$

$$dm = -A^{-2} dA \qquad (2.9)$$

As $m \to \infty;\ \frac{1}{A} \to 0$ and as $m \to 0;\ \frac{1}{A} \to 1$

Substituting Equation (2.9) into Equation (2.8) gives:

$$\int_{-\infty}^{\infty} f(x)\,dx = -\alpha \int_{1}^{0} \left(A^{-1}\right)^{-\alpha-1} A^{-2} dA = \alpha \int_{0}^{1} A^{\alpha-1} dA$$

$$= \alpha \left[\frac{A^{\alpha}}{\alpha}\right]_{0}^{1} = 1^{\alpha} - 0 = 1 \qquad (2.10)$$

The expression in Equation (2.10) indicates that the distribution PDF is a valid probability density function.

2.3 Survival (Reliability) Function

The probability that the item fails after time *x* is denoted by:

$$S(x) = \Pr ob\left(X > x\right) \qquad (2.11)$$

The function *S* is called the survival (reliability) function of *X* defined as:

$$S(x) = 1 - F(x) \qquad (2.12)$$

where, $F(x)$ is the CDF for NBIID in Equation (2.4)

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$$S\left(x\right)=\left\{1-\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-\alpha}\right\} \tag{2.13}$$

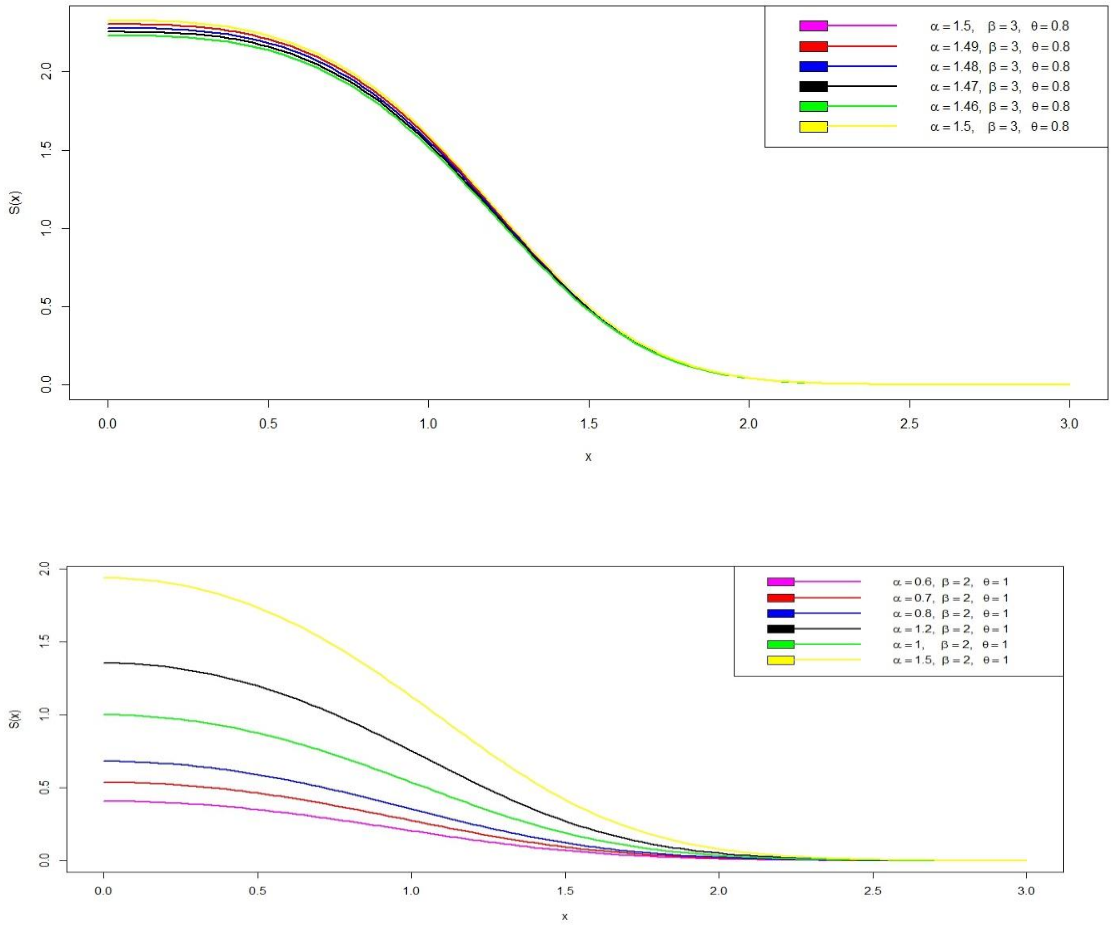

Figure 2. 3: The Survival Plot of NBIID

2.4 Hazard (Failure Rate) Function

The hazard function is the probability that an object will fail or a person will die at a particular time given that the event has not occurred previously. Hazard function is defined as:

$$h_{rf}\left(x\right)=\frac{f\left(x\right)}{S\left(x\right)}=\frac{f\left(x\right)}{1-F\left(x\right)} \tag{2.14}$$

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where, $f(x)$ denotes the PDF of the NBIID in Equation (2.5) and $S(x)$ is the survival function of the NBIID in Equation (2.13)

$$h_{rf}(x)=\frac{\alpha\beta\theta x^{\beta-1}\left(1+x^{\beta}\right)^{\theta-1}e^{1-\left(1+x^{\beta}\right)^{\theta}}\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-(\alpha+1)}}{\left\{1-\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-\alpha}\right\}} \tag{2.15}$$

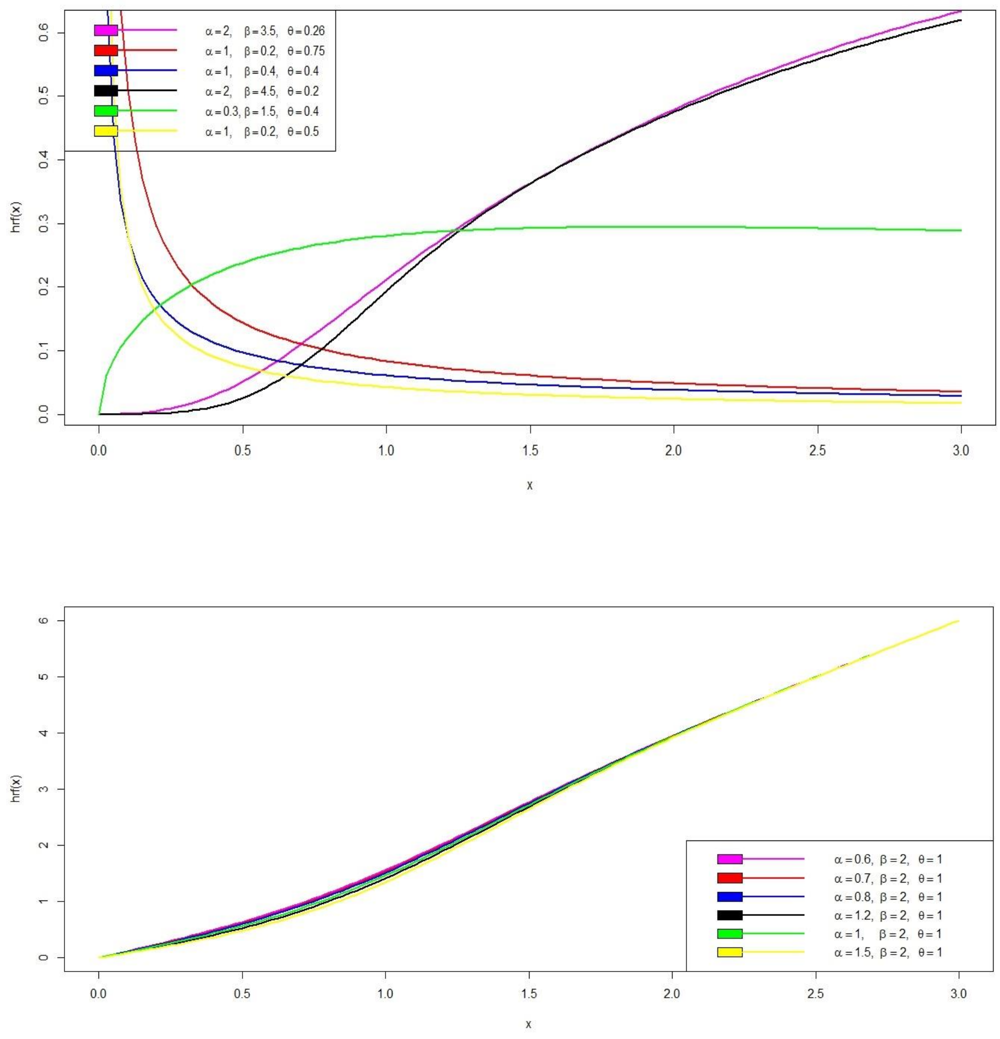

Figure 2. 4: The Hazard Rate Function Plot of NBIID

2.5 Quantile Function

The quantile can be used to generate random numbers. It can also be used to find Quartiles, Median, Octiles, measures of Skewness and Kurtosis. The quantile function of the NBIID is obtained using the CDF as follows

$$F(x)=\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-\alpha}$$

$$\Rightarrow\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]=u^{-\frac{1}{\alpha}} \tag{2.16}$$

$$e^{1-\left(1+x^{\beta}\right)^{\theta}}=\left(u^{-\frac{1}{\alpha}}-1\right)$$

$$1-\left(1+x^{\beta}\right)^{\theta}=\ell n\left(u^{-\frac{1}{\alpha}}-1\right)$$

$$\left(1-\ell n\left(u^{-\frac{1}{\alpha}}-1\right)\right)^{\frac{1}{\theta}}=\left(1+x^{\beta}\right)$$

$$x^{\beta}=\left[1-\left(\ell n\left(u^{-\frac{1}{\alpha}}-1\right)\right)\right]^{\frac{1}{\theta}}-1 \tag{2.17}$$

$$x_u=Q(u)=\left\{\left[1-\left(\ell n\left(u^{-\frac{1}{\alpha}}-1\right)\right)\right]^{\frac{1}{\theta}}-1\right\}^{\frac{1}{\beta}}\ :x_q \tag{2.18}$$

where, u follows a uniform distributed random variable on the interval (0,1). Setting $u=0.5$, the median of NBIID is obtained thus:

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$$x_m = Q_2(0.5) = \left\{ \left[1 - \left(\ell n \left(0.5^{-\frac{1}{\alpha}} - 1 \right) \right) \right]^{\frac{1}{\theta}} - 1 \right\}^{\frac{1}{\beta}} \tag{2.19}$$

The lower quartile and upper quartile can also be obtained from equation (2.18) by respectively setting $u = 0.25$ and $u = 0.75$.

2.6 Order Statistics

In the domains of dependability and life testing, order statistics are extensively utilized. Let $X_{1:n} \le X_{2:n} \le ..., \le X_{n:n}$, be the order statistics obtained from a random sample (r.s) $X_1, X_2, ..., X_n$ from NBIID with the PDF [$f(x)$] and CDF [$F(x)$], respectively, given in Equations (2.5) and (2.4) then the PDF of the mth order statistics, say $X_{i:n} = X_{m:n}$, can be expressed as:

$$f_{m:n}(x) = \frac{n!}{(m-1)!(n-m)!} \left[F(x) \right]^{m-1} \left[1 - F(x) \right]^{n-m} f(x); \; m = 1, 2, ..., n \tag{2.20}$$

Using binomial expansion

$$\left[1 - F(x) \right]^{n-m} = \sum_{i=0}^{n-m} (-1)^i \binom{n-m}{i} \left[F(x) \right]^i \tag{2.21}$$

Substituting (2.21) into (2.20)

$$f_{m:n}(x) = \frac{n!}{(m-1)!(n-m)!} \left[F(x) \right]^{m-1} \sum_{i=0}^{n-m} (-1)^i \binom{n-m}{i} \left[F(x) \right]^i f(x)$$

$$= \sum_{i=0}^{n-m} \frac{(-1)^i n! \cancel{(n-m)}!}{(m-1)! \cancel{(n-m)}! (n-m-i)! i!} \left[F(x) \right]^{m+i-1} f(x)$$

$$= \sum_{i=0}^{n-m} \frac{(-1)^i n!}{(m-1)!(n-m-i)! i!} f(x) \left[F(x) \right]^{m+i-1} \tag{2.22}$$

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Substituting the PDF (*f(x)*) and CDF (*F(x)*) of NBIID into (2.22) above gives the order statistics as:

$$f_{m:n}\left(x\right)=\sum_{i=0}^{n-m}\frac{\left(-1\right)^{i}n!}{i!\left(m-1\right)!\left(n-m-i\right)!}\alpha\beta\theta x^{\beta-1}\left(1+x^{\beta}\right)^{\theta-1}e^{1-\left(1+x^{\beta}\right)^{\theta}}\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-(\alpha+1)}\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-\alpha(m+i)-1} \tag{2.23}$$

The PDF of the smallest order statistics for $m=1$ is given as:

$$f_{1:n}\left(x\right)=n\alpha\beta\theta x^{\beta-1}\left(1+x^{\beta}\right)^{\theta-1}e^{1-\left(1+x^{\beta}\right)^{\theta}}\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-(\alpha+1)}\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-\alpha i-1} \tag{2.24}$$

Furthermore, the PDF of the largest order statistics for $m=n$ is given bellow:

$$f_{n:n}\left(x\right)=n\alpha\beta\theta x^{\beta-1}\left(1+x^{\beta}\right)^{\theta-1}e^{1-\left(1+x^{\beta}\right)^{\theta}}\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-(\alpha+1)}\left[1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right]^{-\alpha n-1} \tag{2.25}$$

2.7 Parameter Estimation

This study proposes two estimation procedures for NBIID, namely MLE and MPS.

2.7.1 Maximum Likelihood Estimation (MLE)

The likelihood function of the developed (NBIID) is given as such that $\left(\phi=\left(\alpha,\beta,\theta\right)\right)$:

$$L\left(\phi\right)=\prod_{i=1}^{n}f\left(x;\alpha,\beta,\theta\right)=\prod_{i=1}^{n}\left[\alpha\beta\theta x^{\beta-1}\left(1+x^{\beta}\right)^{\theta-1}e^{1-\left(1+x^{\beta}\right)^{\theta}}\left(1+e^{1-\left(1+x^{\beta}\right)^{\theta}}\right)^{-(\alpha+1)}\right] \tag{2.26}$$

$$=\alpha^{n}\beta^{n}\theta^{n}\prod_{i=1}^{n}x_i^{\beta-1}\prod_{i=1}^{n}\left(1+x_i^{\beta}\right)^{\theta-1}e^{\sum_{i=1}^{n}\left(1-\left(1+x_i^{\beta}\right)^{\theta}\right)}\prod_{i=1}^{n}\left[1+e^{1-\left(1+x_i^{\beta}\right)^{\theta}}\right]^{-(\alpha+1)} \tag{2.27}$$

and the log-likelihood function is:

$$\ell n\left(L\left(\phi\right)\right)=n\ell n\left(\alpha\right)+n\ell n\left(\beta\right)+n\ell n\left(\theta\right)+\sum_{i=1}^{n}\ell n\left(x_i^{\beta-1}\right)+\sum_{i=1}^{n}\ell n\left(1+x_i^{\beta}\right)^{\theta-1}+\sum_{i=1}^{n}\left(1-\left(1+x_i^{\beta}\right)^{\theta}\right)$$

$$+\sum_{i=1}^{n}\ell n\left[1+e^{1-\left(1+x_i^{\beta}\right)^{\theta}}\right]^{-(\alpha+1)} \tag{2.28}$$

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$$= n\ell n(\alpha) + n\ell n(\beta) + n\ell n(\theta) + (\beta - 1)\sum_{i=1}^{n} \ell n(x_i) + (\theta - 1)\sum_{i=1}^{n} \ell n\left(1 + x_i^{\beta}\right) + n - \sum_{i=1}^{n}\left(1 + x_i^{\beta}\right)^{\theta}$$

$$-(\alpha + 1)\sum_{i=1}^{n} \ell n\left[1 + e^{1-\left(1+x_i^{\beta}\right)^{\theta}}\right] \tag{2.29}$$

Differentiating the log-likelihood function partially with respect to the parameters and equating to zero.

$$\frac{\partial \ell n\left(L(\phi)\right)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \ell n\left[1 + e^{1-\left(1+x_i^{\beta}\right)^{\theta}}\right] = 0 \tag{2.30}$$

$$\frac{\partial \ell n\left(L(\phi)\right)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ell n x_i + (\theta - 1)\sum_{i=1}^{n} \frac{x_i^{\beta} \ell n x_i}{\left(1 + x_i^{\beta}\right)} - \theta \sum_{i=1}^{n} x_i^{\beta}\left(1 + x_i^{\beta}\right)^{\theta - 1} \ell n x_i$$

$$+\theta(\alpha + 1)\sum_{i=1}^{n} \frac{x_i^{\beta}\left(1 + x_i^{\beta}\right)^{\theta - 1} e^{1-\left(1+x_i^{\beta}\right)^{\theta}} \ell n x_i}{\left[1 + e^{1-\left(1+x_i^{\beta}\right)^{\theta}}\right]} = 0 \tag{2.31}$$

$$\frac{\partial \ell n\left(L(\phi)\right)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ell n\left(1 + x_i^{\beta}\right) - \sum_{i=1}^{n}\left(1 + x_i^{\beta}\right)^{\theta} \ell n\left(1 + x_i^{\beta}\right) + (\alpha + 1)\sum_{i=1}^{n} \frac{\left(1 + x_i^{\beta}\right)^{\theta} e^{1-\left(1+x_i^{\beta}\right)^{\theta}} \ell n\left(1 + x_i^{\beta}\right)}{\left[1 + e^{1-\left(1+x_i^{\beta}\right)^{\theta}}\right]} = 0 \tag{2.32}$$

Equations (2.30), (2.31) and (2.32) cannot be solved analytically, thus the use of software like *R* to obtain the parameters.

2.7.2 Maximum Product of Spacing (MPS)

Here, the method of MPS is described briefly as follows:

Supposed $F\left(x_{(i)} \middle| \alpha, \beta, \theta\right)$ and $F\left(x_{(i-1)} \middle| \alpha, \beta, \theta\right)$ for $i = 1, 2, ..., n+1$ are the CDF of the NBIID

Let, $M_i(\alpha, \beta, \theta) = F\left(x_{(i)} \middle| \alpha, \beta, \theta\right) - F\left(x_{(i-1)} \middle| \alpha, \beta, \theta\right)$ (2.33)

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Be the uniform spacing of a random sample generated from the NBIID. Then it can be noted that $F\left(x_{(0)}|\alpha,\beta,\theta\right)=0$ and $F\left(x_{(n+1)}|\alpha,\beta,\theta\right)=1$ such that,

$$\sum_{i=1}^{n+1} M_i\left(\alpha,\beta,\theta\right)=1 \tag{2.34}$$

Therefore, the estimate of the parameters $\left(\alpha,\beta,\theta\right)$ using MPS method are obtained by maximizing the logarithm of Geometric Mean (GM) of spacing as:

$$J\left(\alpha,\beta,\theta\right)=Log\left(GM\left(\alpha,\beta,\theta\right)\right) \tag{2.35}$$

where, $GM\left(\alpha,\beta,\theta\right)=\prod_{i=1}^{n+1} M_i\left(\alpha,\beta,\theta\right)^{\frac{1}{n+1}}$ and

$$J\left(\alpha,\beta,\theta\right)=Log\left(\prod_{i=1}^{n+1}\left[F\left(x_{(i)}|\alpha,\beta,\theta\right)-F\left(x_{(i-1)}|\alpha,\beta,\theta\right)\right]^{\frac{1}{n+1}}\right)$$

$$=\frac{1}{n+1}\sum_{i=1}^{n+1} Log\left[F\left(x_{(i)}|\alpha,\beta,\theta\right)-F\left(x_{(i-1)}|\alpha,\beta,\theta\right)\right] \tag{2.36}$$

For NBIID by inverting Equation (2.4)

$$F\left(x_{(i)}|\alpha,\beta,\theta\right)=\left[1+e^{1-\left(1+x_{(i)}^{\beta}\right)^{\theta}}\right]^{-\alpha}$$

and

$$F\left(x_{(i-1)}|\alpha,\beta,\theta\right)=\left[1+e^{1-\left(1+x_{(i-1)}^{\beta}\right)^{\theta}}\right]^{-\alpha}$$

Thus, maximizing Equation (2.36)

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$$\frac{\partial J(\alpha,\beta,\theta)}{\partial\alpha}=\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{1}{M_i(\alpha,\beta,\theta)}\right)\left[W_1\left(x_{(i)}|\alpha,\beta,\theta\right)-W_1\left(x_{(i-1)}|\alpha,\beta,\theta\right)\right]=0 \tag{2.37}$$

$$\frac{\partial J(\alpha,\beta,\theta)}{\partial\beta}=\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{1}{M_i(\alpha,\beta,\theta)}\right)\left[W_2\left(x_{(i)}|\alpha,\beta,\theta\right)-W_2\left(x_{(i-1)}|\alpha,\beta,\theta\right)\right]=0 \tag{2.38}$$

$$\frac{\partial J(\alpha,\beta,\theta)}{\partial\theta}=\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{1}{M_i(\alpha,\beta,\theta)}\right)\left[W_3\left(x_{(i)}|\alpha,\beta,\theta\right)-W_3\left(x_{(i-1)}|\alpha,\beta,\theta\right)\right]=0 \tag{2.39}$$

where,

$$W_1\left(x_{(i-1)}|\alpha,\beta,\theta\right)=\left[1+e^{1-\left(1+x_{(i-1)}^{\beta}\right)^{\theta}}\right]^{-\alpha}\ell n\left[1+e^{1-\left(1+x_{(i-1)}^{\beta}\right)^{\theta}}\right] \tag{2.40}$$

$$W_1\left(x_{(i)}|\alpha,\beta,\theta\right)=\left[1+e^{1-\left(1+x_{(i)}^{\beta}\right)^{\theta}}\right]^{-\alpha}\ell n\left[1+e^{1-\left(1+x_{(i)}^{\beta}\right)^{\theta}}\right] \tag{2.41}$$

$$W_2\left(x_{(i-1)}|\alpha,\beta,\theta\right)=\alpha\theta\left[1+e^{1-\left(1+x_{(i-1)}^{\beta}\right)^{\theta}}\right]^{-(\alpha+1)}e^{1-\left(1+x_{(i-1)}^{\beta}\right)^{\theta}}\left(1+x_{(i-1)}^{\beta}\right)^{\theta-1}x_{(i-1)}^{\beta}\ell n x_{(i-1)} \tag{2.42}$$

$$W_2\left(x_{(i)}|\alpha,\beta,\theta\right)=\alpha\theta\left[1+e^{1-\left(1+x_{(i)}^{\beta}\right)}\right]^{-(\alpha+1)}e^{1-\left(1+x_{(i)}^{\beta}\right)^{\theta}}\left(1+x_{(i)}^{\beta}\right)^{\theta-1}x_{(i)}^{\beta}\ell n x_{(i)} \tag{2.43}$$

$$W_3\left(x_{(i-1)}|\alpha,\beta,\theta\right)=\alpha\left[1+e^{1-\left(1+x_{(i-1)}^{\beta}\right)^{\theta}}\right]^{-(\alpha+1)}e^{1-\left(1+x_{(i-1)}^{\beta}\right)^{\theta}}\left(1+x_{(i-1)}^{\beta}\right)^{\theta}\ell n\left(1+x_{(i-1)}^{\beta}\right) \tag{2.44}$$

$$W_3\left(x_{(i)}|\alpha,\beta,\theta\right)=\alpha\left[1+e^{1-\left(1+x_{(i)}^{\beta}\right)^{\theta}}\right]^{-(\alpha+1)}e^{1-\left(1+x_{(i)}^{\beta}\right)^{\theta}}\left(1+x_{(i)}^{\beta}\right)^{\theta}\ell n\left(1+x_{(i)}^{\beta}\right) \tag{2.45}$$

The solution to Equations (2.37), (2.38) and (2.39) are obtained using numerical solution found in software like R and Python.

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2.8 Evaluation of Efficiency and Consistency of Parameter Estimation Methods

Simulated datasets with specific parameter values and increasing sample sizes were generated to check for consistency and efficiency of the parameter estimation methods (MLE and MPS) proposed for NBIID.

i. Consistency is established if the estimated parameter values converge towards the specific parameter values as sample sizes increases. The estimated parameter values are not to change substantially with increasing sample sizes.

ii. Mean square errors were computed for specific parameter values at increasing sample sizes. A lower Mean Square Error (MSE) indicates a more efficient estimator, the estimation method that performs better.

iii. Simulation Studies: Simulation was repeated for N = 1000 replications based on the quantile function defined in Equation (2.18). Sample sizes of sizes n = 10, 20, 30, 50, and 100 were considered for the following parameter combinations: $\alpha = 1, \beta = 1, \theta = 1$; $\alpha = 1.5, \beta = 1, \theta = 1$; and $\alpha = 2, \beta = 1, \theta = 1$.

a. Generate $u_i \sim U(0, 1)$, $i = 1, 2, ..., n$;

b. Determine the random samples from the NBII distribution

c. Apply MLE and MPS estimation methos

d. Repeat steps a to c for 1000 replications to obtain the estimates

e. Compute bias and MSE to investigate the precision of the MLEs and MPS as follows:

$$Bias = \frac{1}{1000}\sum_{i=1}^{1000}\left(\phi - \phi\right) \text{ and } MSE = \frac{1}{1000}\sum_{i=1}^{1000}\left(\phi - \phi\right)^2, \text{ where } \phi = \left(\alpha, \beta, \theta\right)^T$$

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2.9 Performance Evaluation of NBIID

NBIID was applied to four real-life data sets and compared to two competing distributions using AIC, BIC, CAIC), and HQIC. A lower performance metric value indicates better performance.

3. Results and Discussion

3.1 Simulation Study

Simulation study was conducted to evaluate the adaptability, consistency and efficiency of the parameter estimators of NBIID.

The MLEs and MPS accuracy were evaluated for various sample sizes and specific parameter combinations. The 1000 replications of the simulations were carried out in order to assess the effectiveness of MLE and MPS figures. The study computed a number of statistics, including the mean, biases, and mean squared errors (MSEs), based on three different parameter combinations as indicated in Table 3.1, Table 3.2 and Table 3.3 below.

Table 3. 1: Performance Evaluation of MLE and MPS Methods of Estimation

		MLE			MPS		
n	**Parameters**	**Estimate**	**Bias**	**MSE**	**Estimate**	**Bias**	**MSE**
10	α=1	1.0600	0.0600	0.0685	1.0406	0.0406	0.0251
	β=1	0.9996	-0.0004	0.0001	1.0001	0.0001	0.0003
	θ=1	1.0017	0.0017	0.0006	1.0007	0.0007	0.0001
20	α=1	1.0441	0.0441	0.0041	1.0314	0.0314	0.0029
	β=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	θ=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
30	α=1	1.0414	0.0414	0.0038	1.0299	0.0299	0.0027
	β=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	θ=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
50	α=1	1.0379	0.0379	0.0034	1.0287	0.0287	0.0026
	β=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	θ=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
100	α=1	1.0328	0.0328	0.0027	1.0250	0.0250	0.0020
	β=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	θ=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000

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Table 3. 2: Performance Evaluation of MLE and MPS Methods of Estimation

n	Parameters	MLE			MPS		
		Estimate	Bias	MSE	Estimate	Bias	MSE
10	α=1.5	1.6210	0.1210	0.3385	1.5879	0.0879	0.1874
	β=1	0.998	-0.002	0.0063	0.9969	-0.0031	0.0044
	θ=1	1.0132	0.0132	0.0046	1.0082	0.0082	0.0026
20	α=1.5	1.5683	0.0683	0.0182	1.5513	0.0513	0.0298
	β=1	0.9997	-0.0003	0.0001	0.9995	-0.0005	0.0002
	θ=1	1.0013	0.0013	0.0003	1.0011	0.0011	0.0004
30	α=1.5	1.5619	0.0619	0.0085	1.5448	0.0448	0.0060
	β=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	θ=1	1.0002	0.0002	0.0000	1.0001	0.0001	0.0000
50	α=1.5	1.5568	0.0568	0.0076	1.5431	0.0431	0.0057
	β=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	θ=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
100	α=1.5	1.5492	0.0492	0.006	1.5375	0.0375	0.0045
	β=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	θ=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000

Table 3. 3: Performance Evaluation of MLE and MPS Methods of Estimation

n	Parameters	MLE			MPS		
		Estimate	Bias	MSE	Estimate	Bias	MSE
10	α=2	2.2975	0.2975	1.3221	2.1972	0.1972	0.6137
	β=1	0.9916	-0.0084	0.0320	0.9918	-0.0082	0.0150
	θ=1	1.0472	0.0472	0.0198	1.0277	0.0277	0.0112
20	α=2	2.0998	0.0998	0.1098	2.0734	0.0734	0.0793
	β=1	0.9984	-0.0016	0.0006	0.9984	-0.0016	0.0006
	θ=1	1.0110	0.0110	0.0020	1.0070	0.0070	0.0013
30	α=2	2.0802	0.0802	0.0145	2.0589	0.0589	0.0105
	β=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	θ=1	1.0032	0.0032	0.0003	1.0019	0.0019	0.0002
50	α=2	2.0752	0.0752	0.0133	2.0572	0.0572	0.0101
	β=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	θ=1	1.0005	0.0005	0.0000	1.0004	0.0004	0.0000
100	α=2	2.0656	0.0656	0.0107	2.0500	0.0500	0.0079
	β=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
	θ=1	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000

Table 3.1 presents the estimates of the parameters using MLE and MPS methods with $\alpha = \beta = \theta = 1$. As seen from Table 3.1, for the increasing sample sizes of $n = 10,\ 20,\ 30,\ 50$ and 100, the mean of each estimate using the methods of MLE and

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MPS of estimation approaches true parameter values. Similarly, the MSE of each estimate using MLE and MPS of estimation decreases and approaches zeros. The table reveals that with the increasing sample sizes, both MLE and MPS methods yield similar results with MPS producing lower MSE compared to MLE.

Table 3.2 provides the estimates of the parameters in which $\alpha = 1.5,\ \beta = 1,\ \theta = 1$. Table 3.2 reveals that the mean estimates of each parameter using MLE and MPS methods of estimation approach fixed parameter value $\alpha = 1.5,\ \beta = 1,\ \theta=1$, as the sample sizes increases. The MSE of the parameters using MLE and MPS methods of estimation decreases and converge to zero. It also proves that the MSE using MLE and MPS still approaches similar result as the sample sizes increases. Likewise, MPS provides the least MSE compared with MLE.

Table 3.3 shows that the mean estimates of each parameter using MLE and MPS of estimation approach fixed parameter value $\alpha = 2,\ \beta = 1,\ \theta = 1$. As sample sizes increase, the MSE of the parameter using MLE and MPS methods of estimation decreases and converges to zero. Hence, MPS provides the least MSE compared to MLE, this indicates better estimation in comparison with MLE.

3.2 Application to Real-life Data Sets

This section presents application of NBIID to real-world data sets in order to determine their potential and performance in comparison to other competing distributions. The unbounded ($-\infty, +\infty$) distributions, namely the Transmuted Burr Type II (TBIID) and the base distribution, the Burr Type II distribution, distributions were considered for comparison of performance.

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3.2.1 Data 1

The data represents the life of fatigue fracture of Kevlar 373/epoxy subjected to constant pressure at 90% stress level until all had failed. The data has also been used by David et al. (2021), Mohammed and Ugwuowo (2020), Owoloko et al. (2015), among others. The data is as follows: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

3.2.2 Data II

The second data set is a subset of the data reported by Bekker et al. (2000), which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. The data consist of survival times (in years) for 45 patients. It is as follows: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

4.2.3 Data III

The third data set is the survival times (in days) of 72 guinea-pigs infected with virulent tubercle bacilli. This data set was originally discussed and reported by Bjerkedal (1960) and recently by Chhetri et al. (2022) and Mohammed et al. (2023). For computational

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convenience, each lifetime point in the original guinea-pigs data set is divided by one hundred. New transformed survival times of 72 guinea-pig is as shown: 0.10, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1.00, 1.00, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.20, 1.21, 1.22, 1.22, 1.24, 1.30, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.60, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.30, 2.31, 2.40, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55

4.2.4 Data IV

The considered data set represents the maximum levels of the flood used by Alshanbari et al. (2023).

0.7791985, 1.1483097, 2.9672206, 1.4938204, 1.5833982, 1.7602758, 0.2158896, 0.3316141, 0.9555678, 0.6043510, 0.8510060, 0.4920070, 0.3715561, 0.6607851, 0.4876386, 0.4834433, 0.7375517, 0.5111949, 1.1339544, 0.3851595, 2.3691135, 0.5186865, 0.4137324, 0.7189331, 0.7381207, 0.3792905, 0.3972784, 0.6167996, 0.8184564, 0.3787097, 2.0782179, 1.4738027, 2.2733590, 1.4995140, 2.3685689, 0.7929649, 1.3452230, 3.7218931, 0.4847362, 0.5570308, 0.6421779, 0.3521637, 1.2319488, 1.0085401, 0.5739284, 0.8282354, 1.3438912, 0.7985639, 0.5913791, 0.3593687, 0.6421779, 0.3521637, 1.2319488, 1.0085401, 0.5739284, 0.8282354, 1.3438912, 0.7985639, 0.5913791, 0.3593687, 1.2470297, 1.5119436, 1.2313264, 0.4174427, 0.4856877, 0.7467844, 1.4316524, 2.0115124, 0.5920552, 0.6354470, 0.2082874, 0.8970404, 0.4521464, 0.8233669, 1.0459246, 0.8731764, 0.8302978, 1.5595648, 0.8808601, 1.5319633, 0.7943833, 1.0328066, 2.1259828, 0.6339371, 1.3113389, 0.8507588, 1.1216619, 4.5418208, 1.2421274, 3.5577773, 2.0439183, 0.3183116, 0.7603422, 0.6969910, 0.6208522, 0.7419367, 2.3501271, 0.4923549,

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0.9713638, 1.7159552, 1.1431615, 0.5176597, 0.6126890, 0.3778499, 0.4487731, 0.4412745, 0.2943709, 0.2843115, 0.3939066, 0.4730012, 1.8675487, 1.6653246, 1.9875657, 1.7238783, 1.8093874, 1.9698237, 1.8123847, 1.9654765, 1.8654765, 2.1987234, 2.2165476, 2.8654765, 2.9216547, 2.8012938, 2.8092834, 2.7165476, 2.8165476, 3.2654765, 3.2912873, 3.2091827, 3.2234893, 3.6547652, 3.9654765, 3.2187324, 4.9827360, 4.1372098, 4.1412307, 5.1987322, 5.1988294, 6.1876232, 5.9826342.

Table 3. 4: **Descriptive statistics for the data sets**

Statistics	Data 1	Data 2	Data 3	Data 4
Observations	76	45	72	141
Min	0.0251	0.047	0.100	0.2083
Max	9.096	4.033	5.550	6.1876
Mean	1.9592	1.3414	1.7682	1.4706
Std. Dev	1.5739	1.2466	1.0345	1.2614
1st Quartile	0.9048	0.395	1.080	0.5914
Median	1.7362	0.841	1.495	0.9714
3rd Quartile	2.2959	2.178	2.240	1.9698
Skewness	1.9406	0.9399	1.3140	1.5943
Kurtosis	4.9474	-0.4532	1.8534	2.2631

The summary of the data set, including mean, standard deviation, skewness and kurtosis, is provided in Table 3.4. It can be observed from the table that the skewness measure of the real-life data sets is positive. So, the distribution of the data is skewed to the right. Also, since the kurtosis measure of data 1 is greater than 3 (> 3), the distribution is leptokurtic in nature. For data 2, data 3, and data 4, the kurtosis measure is less than 3 (< 3), so the distribution for the data is platykurtic in nature.

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Table 3. 5: Performance of the NBII Distribution against Competing Models using Data 1

Model	Estimates	L	AIC	BIC	CAIC	HQIC
NBIID	α = 8.8667 (2.5878) β = 0.5280 (0.1092) θ = 1.5247 (0.1708)	**120.9115**	**247.823**	**254.8152**	**248.1563**	**250.6174**
TBIID	α = 5.5122 (0.2114) β = 0.4454 (0.6990)	125.4005	254.801	259.4624	254.9654	256.6639
BIID	α = 4.5349 (0.5201)	126.765	255.5301	257.8608	255.5841	256.4615

Table 3. 6: Performance of the NBII Distribution against Competing Models using Data 2

Model	Estimates	L	AIC	BIC	CAIC	HQIC
NBIID	α = 15.9096 (13.1338) β = 0.2579 (0.1047) θ = 2.1037 (0.3129)	**58.7088**	**123.4178**	**128.8377**	**124.0031**	**125.4383**
TBIID	α = 3.7389 (0.5718) β = 0.5287 (0.2478)	69.4733	142.9467	146.5601	143.2325	144.2938
BIID	α = 2.9968 (0.4467)	70.991	143.9821	145.7888	144.0751	144.6556

Table 3. 7: Performance of the NBII distribution against Competing Models using Data 3

Model	Estimates	L	AIC	BIC	CAIC	HQIC
NBIID	α = 12.7231 (4.8749) β = 0.6060 (0.1389) θ = 1.6205 (0.2001)	**94.1345**	**194.2689**	**201.1403**	**194.6167**	**197.0073**
TBIID	α = 6.6916 (0.6949) β = 0.8254 (0.1509)	97.1645	198.3291	202.91	198.5005	200.1547
BIID	α = 4.7871 (0.5603)	103.1977	208.3953	210.6858	208.4517	209.3081

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Table 3. 8: Performance of the NBII Distribution against Competing Models using Data 4

Model	Estimates	L	AIC	BIC	CAIC	HQIC
NBIID	α = 13.2566 (3.2391) β = 0.3830 (0.0476) θ = 1.9264 (0.1065)	**181.5583**	**369.1166**	**377.9629**	**369.2918**	**372.7114**
TBIID	α = 4.4485 (0.3544) β = 0.6435 (0.1219)	208.9517	421.9033	427.8009	421.9903	424.2999
BIID	α = 3.4109 (0.2872)	216.6811	435.3622	438.3109	435.391	436.5605

The performances for NBIID and other competing distributions with applications to real-life data sets 1, 2, 3 and 4 are given in Tables 3.5, 3.6, 3.7, and 3.8. The NBIID gives the least values of AIC, BIC, CAIC, and HQIC statistics compared to other comparative distributions. This shows that the new distribution performs better for the datasets under study.

The density plots for NBIID against its comparative distributions using Data 1, 2, 3 and 4 are provided in Figures 3.1, 3.2, 3.3, and 3.4.

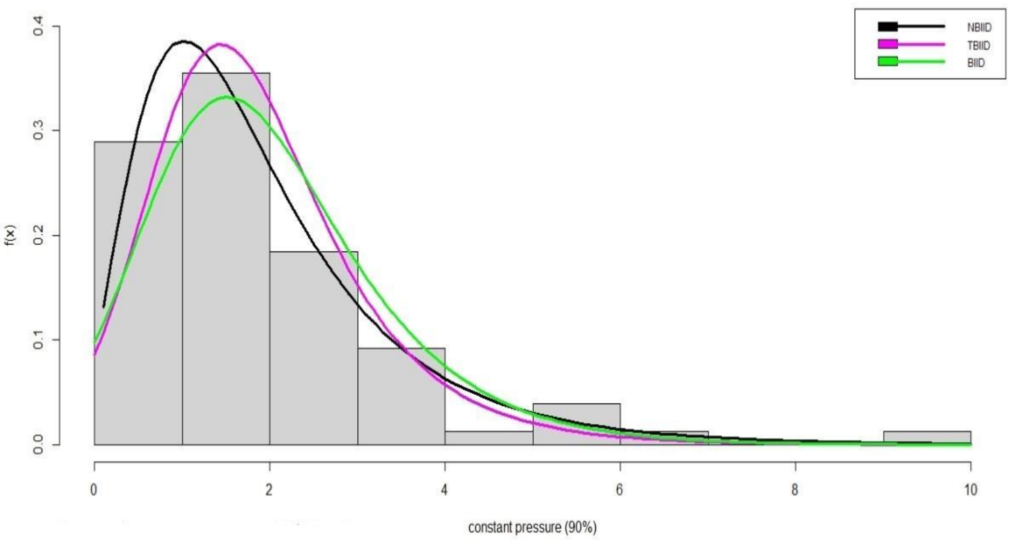

Figure 3. 1: The Fitted PDF for Fatigue Fracture Data 1

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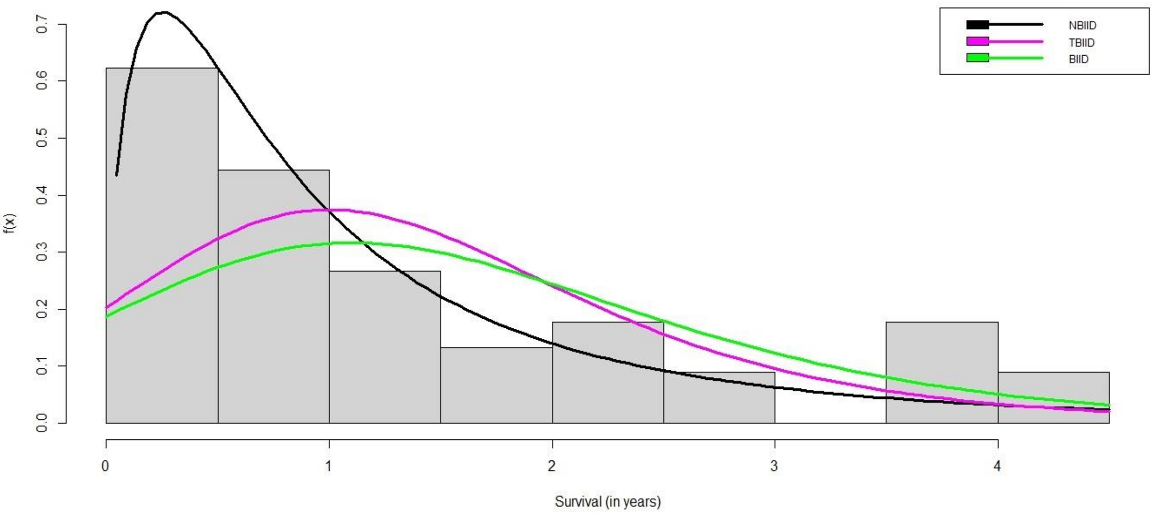

Figure 3. 2: The Fitted PDF for Chemotherapy Patient Data 2

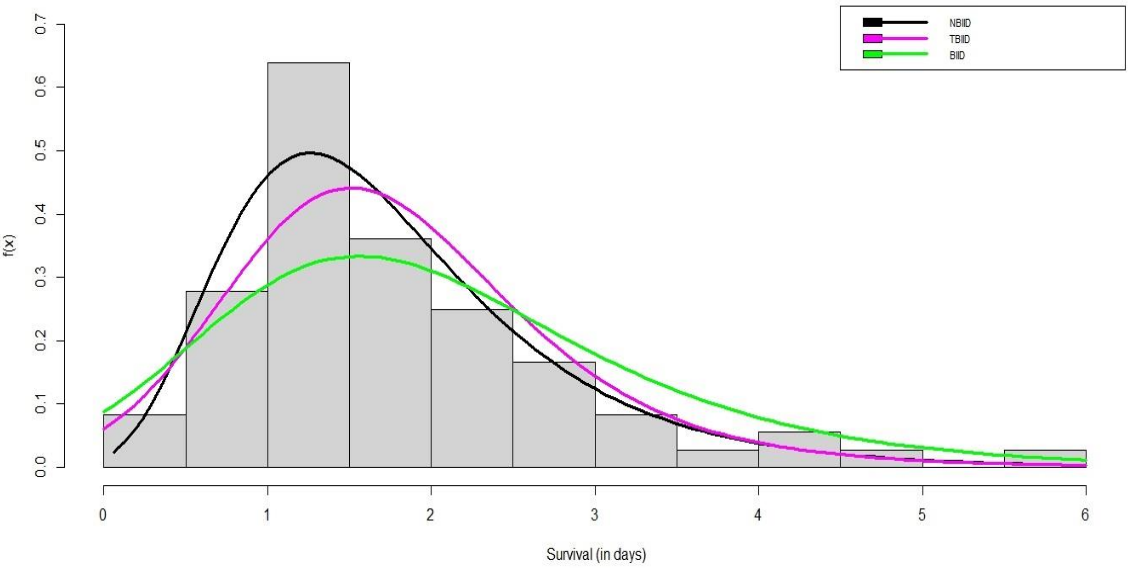

Figure 3 3: The Fitted PDF for Guinea-pigs' Data 3

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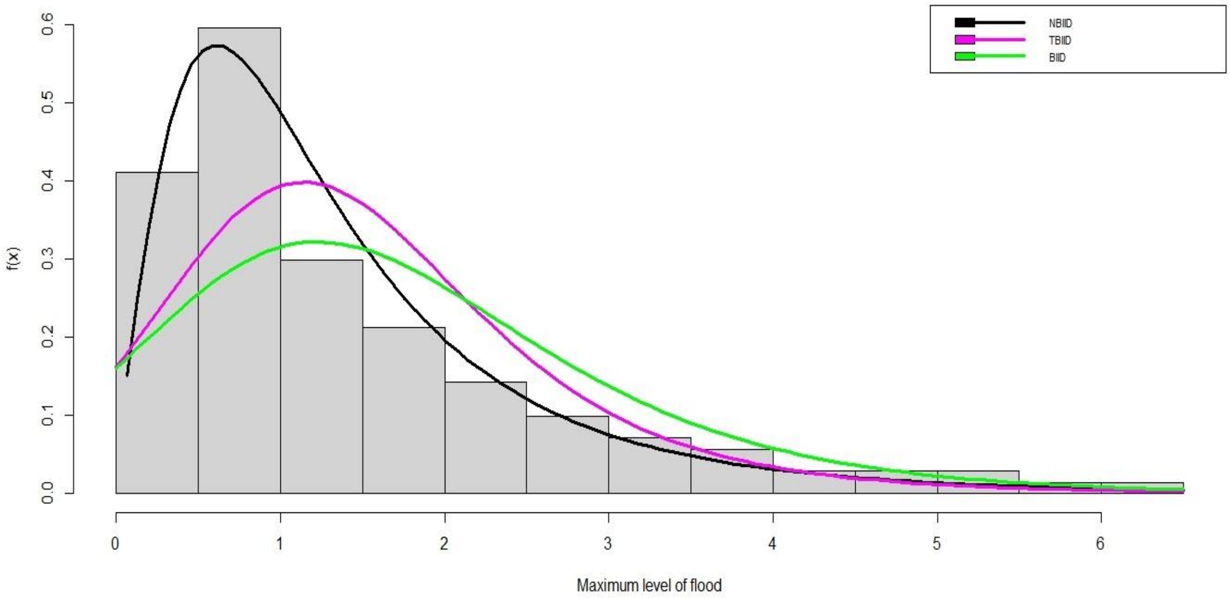

Figure 3 4: The Fitted PDF for Flood Data 4

4. Conclusion

Studies show that adding parameter(s) to existing distributions increases flexibility and goodness of fit, making such distributions suitable for modeling variety of data types The Burr type II distribution was modified to derive a three-parameter distribution termed the New Burr type II Distribution. Properties of the new distribution were studied. The density shape is unimodal and could be approximately symmetric and skewed. Some important features of the distribution were identified. The model parameters were estimated using MLE and MPS estimation methods. MPS was established to perform better with a lower MSE. Further studies will serve to reveal other interesting properties of the NBIID. NBIID is hence proposed a good alternative for modelling continuous real-life data sets.

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