Profiling the Optimality of Connected Incomplete Block Designs of Size (t=9, b=9, k=3, r=3) Through Their Adjacency Matrices.

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ABSTRACT

This article extended the six (6) Incomplete Block Designs (IBDs) represented by letters; *A, B, C, D, E* and *F* of size $(t = 9, b = 9, k = 3, r = 3)$ proposed by Nguyen, (1994), and included the combinatorial property of design adjacency on them by constructing seventeen (17) additional IBDs namely; G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V and W using all the initial blocks in Nguyen's via the cyclic method and JAVA codes. The concurrence matrices of all the emerging twenty-three (23) IBDs were subjected to combinatorial conformity via their adjacency matrices based on the numbers of "*0*'s" and "*1*'s" in the design layouts, which indicates the extent of connectedness of the designs. Designs H, J, N, O, P, Q, and W turned out to be the preferred set of designs based on the design adjacency, as they provided the best choice of treatment combinations in blocks. Therefore, any of the seven identified optimal designs could be of use by experimenters, investigators, or evaluators who seek an efficient design of size $(t = 9, b = 9, k = 3, r = 3)$ in any field experiment, with their experimental layouts, especially when adjacency of blocks as a combinatorial property is of interest to the researcher.

Key Words: Combinatorial, Connectedness, Adjacency, Constructing, Optimal designs, Efficient design.

1.0 INTRODUCTION

Nguyen (1994) discovered and utilized a set of six IBDs of size $(t = 9, b = 9, k = 3, r = 3)$ constructed with treatments greater than 14 in all the examples that motivated his construction. The constructed designs could be used for experimentation that involved the removal of lichens from asbestos cement (AC), for testing the weathering stability of some brands of paints under ultraviolet (UV) irradiation, where the UV weather meter could only hold three paint panels at a time. However, in illustrating the steps of his algorithm for the construction of the IBD, he did not involve the use of optimality criteria such as the -D and -E, and other combinatorial properties such as the design connectedness/circuits via concurrence graphs, adjacency matrices of the optimal or near optimal designs, and efficiency factors of the design. There is a need to expand the research to include more combinatorial properties, such as the establishment of concurrence graphs. According to Haemers (2011), the adjacency matrix of a graph can be seen as the incidence matrix of a design or as the generator matrix of a binary code. The author emphasized that these relationships are crucial and examined graphs for which the corresponding design is a (symmetric) block design or a (group) divisible design. It was suggested that such graphs are strongly regular in the case of a block design, or very similar to a strongly regular graph in the case of a divisible design. The research included constructions and properties related to these types of graphs, which were considered as the binary code of a strongly regular graph; some theories were worked out, and several examples were given.

Giampiero et al. (2017) conducted an important study of graphs through the lens of granular computation (GrC), interpreting the adjacency matrix of any simple undirected graph (G) as a data information table—a structure that holds considerable significance in database theory. Their research establishes a robust mathematical connection between GrC and graph theory. The authors effectively demonstrate that the well-known concept of the indiscernibility relation in Generalized Relation Composition (GrC) functions as a symmetry relation concerning any selected subset of vertices in graph theory. They carefully analyzed a simple undirected graph (G) by choosing a generic vertex subset as a reference system. This approach allowed for an examination of the symmetry of all vertex subsets of (G). The transition from viewing (G) without a reference system to considering the pair (G, W) reflects the important shift from an affine space to a vector space. Furthermore, the authors skillfully interpreted the symmetry blocks within the reference system (G,W) as specific equivalence classes of vertices in (G). They thoroughly investigated the geometric properties of all reference systems (G,W) as the subset (W) varied across all vertex combinations of (G). Additionally, they introduced three innovative hypergraph models and a vertex set partition lattice associated with (G), grounding their framework in several classical notions of GrC. Their comprehensive geometric characterization and structural analysis of these constructs for fundamental graph families highlight the significance of their findings.

Chigbu and Ekpo (2010) also presented results regarding the adjacency of the A-, D- , and E-optimal $(4 \times 4)/4$ Semi-Latin squares due to Bailey and Chigbu (1997); the most connected design among the three for experimentation was discriminated based on the numbers of “0’s” and “1’s” in the adjacency matrix of each of the design. Morgan and Wang (2010) developed an optimality framework for designing experiments in which not all treatments are of equal interest, particularly those having an established control.

In this study, we meticulously crafted an impressive total of seventeen Incomplete Block Designs (IBD) of the dimensions (9, 9, 3, 3). This was achieved by skillfully leveraging all the foundational blocks laid out in Nguyen’s innovative framework. Each design reflects a careful consideration of structure and balance, showcasing the versatility and potential of IBDs in our research endeavors. This process was executed using a cyclic method implemented through Java programming. The study further evaluates the degree of connectedness among all designs by analyzing the adjacency of their treatment combinations within the blocks. Furthermore, it identifies the most effective designs for experimentation based on the newly constructed designs and assesses their respective levels of connectedness.

2.0 LITERATURE REVIEW

This article asserts the importance of understanding the connectedness of incomplete block designs, particularly in terms of their adjacency. We thoroughly examined relevant literature that provides valuable historical and conceptual insights into incomplete block designs and their adjacency.

Bapat and Roy (2013) demonstrated significant findings regarding the adjacency matrices over IF2 of claw-free block graphs, which are equivalent to the line graphs of trees. Additionally, they analyzed the adjacency matrices of flowers, which are defined as block graphs containing only one cut-vertex.es of flowers, which are block graphs with only one cut-vertex.

W. H. Haemers (2011) articulates that the adjacency matrix of a graph is fundamentally interpretable as both the incidence matrix of a design and the generator matrix of a binary code. Furthermore, the role of relations in this context is crucial. Haemers rigorously investigates graphs associated with designs classified as symmetric block designs or group divisible designs. He concludes that these graphs demonstrate strong regularity in the case of block designs and reveal substantial similarities to strongly regular graphs when it comes to divisible designs.

Wich Moonsarn and Jeffrey Zhang (2024), in their Special Mathematics Lecture (Graph Theory) delivered at Nagoya University, Spring 2024, opined that the Incidence matrix and Adjacency matrix are two different ways to store a graph as a matrix. Some matrices might be easier to analyze in some cases than others. They proposed an algorithm to transform an Incidence matrix into an Adjacency matrix for an undirected graph.

Paramadevan, P, and Sotheeswaran, S (2021) examined the adjacency matrix of a graph, claiming it to be the sole fundamental matrix linked to any graph, and concluded that the characteristics of the adjacency matrix and its related theorems are essential since graphs are stored in computers as their adjacency matrix.

Yilmaz and Durmas (2012) noted that there has been growing interest in graph theory recently, alongside intensive research into calculating integer powers of matrices. In their study, they focused on a specific type of directed graph and derived a general representation of the adjacency matrices for that graph, utilizing the well-known property that states the entry at position (i, j) in Am (where A denotes the adjacency matrix) indicates the number of walks of length m from vertex i to vertex j. They also shed light on the elements of the mth positive integer power of the adjacency matrix that are associated with well-known Jacobsthal numbers. As a result, they presented a Cassini-like formula for Jacobsthal numbers and a matrix whose Jacobsthal numbers equate to permanents.

Crnković, D et al. (2024), in their research, constructed distance-regular graphs that exhibit a vertex transitive action from the five sporadic simple groups identified by E. Mathieu, namely the Mathieu groups M11, M12, M22, M23, and M24. Through the binary code formed by the adjacency matrix of a strongly regular graph with parameters (176, 70, 18, 34), they derived block designs whose full automorphism groups are isomorphic to the Higman-Sims finite simple group.

Vasiliki K. et. al (2020) stated that in addition to the adjacency matrix, a graph G can also be depicted using other connectivity matrices such as the Laplacian matrices. They highlighted that the primary differences among spectral clustering methods arise from the selection of the Laplacian matrix and argued that utilizing the normalized Laplacian graph, where the entries represent the degrees of the vertices, partially provides the transition matrix of a standard random walk on the graph, thus serving as a valuable tool for capturing a diffusion process, like treatment propagation effects, within a network. They recalled that a walk is a sequence of edges linked to a sequence of vertices, wherein vertices may recur, while a random walk is a type of walk across a network accomplished by taking successive random steps. They concluded that vertices with a high degree are more likely to be encountered by the random walk because there are more potential paths leading to them.

3.0 METHOD

The article presents a method for constructing additional Incomplete Block Designs (IBDs) of size (9, 9, 3, 3) by utilizing all the initial blocks from Nguyen (1994) through a cyclic approach implemented via Java code. Prior to this research, Nguyen (1994) had established a set of six optimal incomplete block designs, labeled A, B, C, D, E, and F, which were identified as a similar set, as outlined in Appendix 1. Using the cyclic method, the Java code generated seventeen additional incomplete block designs of the same class and size, which have been designated as designs G through W. As a result, this research produced a comprehensive set of twenty-three IBDs. These designs were evaluated for their suitability for field experimentation, particularly focusing on the adjacency of treatment combinations within the designs. The adjacency matrix of a design, along with its concurrence graph, can be interpreted as the incidence matrix of the design or as the generator matrix of a binary code. In practical experimental design, it is crucial to ensure precision and to have a clear understanding of the number of treatments involved. Thus, when selecting a balanced IBD, it is essential to meet precision requirements by constructing various IBDs for any specified treatment, and choosing those that fulfill the precision criteria with a minimal indicator such that:

$$N = tr \tag{1.0}$$

where "N" is the incidence matrix, "t" the treatments and "r" the replication.

According to Dieter and Gunter (1986), a balanced IBD with the parameters $t, b, r^*, k^*, \lambda^*$, and n can be obtained by extending the blocks of an n-fold complete block design by a Balanced Incomplete Block (BIB) design with the parameters:

$$t, b, r = r^* - nb \tag{2.0}$$

$$k = k^* - nt \tag{3.0}$$

and $$\lambda = \lambda^* - 2rn - n^2 b \tag{4.0}$$

Where "t" is the treatment, "b" the block, "r" the replication, "r^*" another class of replication, "n" the number of treatments, k-plots, k^* is another class of k-plots, λ is the number of times pairs of treatments appear together while λ^* is another number of times a pairs of treatments appear together.

There are multiple methods to create a balanced IBD, and these should be employed when the parameters are significantly large. Designs for smaller values can be developed through trial and error without excessive difficulty (Bose, 2011). Additionally, constructing a balanced IBD relies on the idea of symmetrically repeated differences. By applying this principle, it becomes feasible to build the entire design using the initial blocks. The process of identifying the initial blocks is greatly aided by leveraging the characteristics of the primitive roots of binomial equations within the Galois field (Mahanta, 2018). When the initial blocks can be established through trial or other means, and while meeting the condition of symmetrically repeated differences, it is possible to create a design (Bose, 2011). However, the selection of the initial block is somewhat subjective, as it can lead to the formation of an appropriate design using a cyclical substitution method. For simplicity, the initial block will be chosen as the one with the smallest numerical values. For instance, consider an IBD of a specific size, and using a block as the initial block will result in the following seven incomplete blocks: (0 1 3), (1 2 4), (2 3 5), (3 4 6), (4 5 0), (5 6 1), (6 0 2). In this case, treatments are replicated three times, with each pair of treatments appearing together only once in a block. The cyclic construction method involves obtaining a block by simply adding one to each element of the previous block and applying modulo 7 when required. The following theorems illustrate how to construct IBDs using this cyclic approach: Given an initial block after labeling the treatments by modulo, the other blocks of the cyclic design follow, with all calculations performed in modulo t. This cyclic construction method is applicable when b equals t. Each selection of an initial block has an accompanying difference table, similar to the one created below:

$$\begin{array}{c|cccc} & i_1 & i_2 & \cdots & i_k \\ \hline i_1 & 0 & i_2 - i_1 & \cdots & i_k - i_1 \\ i_2 & i_1 - i_2 & 0 & \cdots & i_k - i_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ i_k & i_1 - i_k & i_2 - i_k & \cdots & 0 \end{array}$$

The cyclic method of constructing designs involves generating a new block which entails adding the number "1" to each element of the previous block while applying the appropriate modulo when necessary (John, 1987; Bailey and Cameron, 2019). The cyclic method is employed in this study, particularly to create seventeen new incomplete block designs (IBDs) of the same size (9, 9, 3, 3) as those established by Nguyen (1994). These designs are designated as designs G through W, respectively. Specifically, design G, was constructed using Nguyen's initial block (3, 1, 7), while design H was based on the initial block (3, 1, 9). Design I utilized the initial block (9, 8, 4), design J used (7, 8, 4), design K was formulated from (5, 2, 6), and design L from (3, 2, 7). Similarly, design M was constructed using the block (9, 2, 7), design N with (1, 2, 7), design O with (5, 9, 8), design P with (5, 3, 8), design Q with (6, 1, 4), design R with (5, 9, 4), design S with (2, 8, 3), design T with (2, 8, 9), design U with (2, 8, 1), design V with (7, 1, 6), and lastly, design W with (7, 3, 6). The newly constructed incomplete block designs can be found in Appendix 2, along with their corresponding outputs in Appendix 3.

In a connected design, all treatment contrasts within a block are estimable, and the variances of pairwise comparisons of estimators are similar. Given two treatment effects, τ_{i1} and τ_{i2} it is possible to have treatment effects arranged in chains, $\tau_{i1}, \tau_{1j}, \tau_{2j}, \ldots, \tau_{nj}, \tau_{i2}$ such that two adjoining treatments within this chain occur in the same block. A design is considered connected if you can access any treatment from any other treatment. This definition is supported by Toutenburg and Shalabh (2009).
To better understand the concept of connectedness, we can create an incidence matrix:

$$N = (n_{ij}) \qquad (5.0)$$

Here, N is denoted by $(t \times b)$matrix with n_{ij} being the number of times that the i^{th} treatment within the blocks, appears in the j^{th} block of the design. However, the equality will hold if and only if the design is connected, particularly in a binary design whereby each element in the incidence matrix, N is either "0" or "1" (Onukogu and Chigbu, 2002). The adjacency matrix, P, which is also called the Hamiltonian operator $P = \lfloor P_{ij} \rfloor$of a graph G having "n" vertices in a square $(n \times n)$ symmetric matrix that contain valid information about the internal connectivity status of vertices in the graph, is such that the number 1 is recorded if vertices i and j are connected and 0, otherwise. Suppose there is an edge from vertex $i - j$, $P_{ij} = 1$and 0 if there is no edge from vertex $i - j$. However, a matrix with only "0's" and "1's" as entries is called a $(0, 1) - \text{matrix}$ with $P_{ij} = P_{ji} = 1$ if and only if the vertices i and j are joined. Clearly, P is symmetric with zeros on the diagonal entries (Lint and Wilson, 2011). The adjacency matrix is a simple matrix with elements "0's" and "1's" signifying adjacency in treatment combination, but for the undirected graphs, the adjacency matrix is symmetric (Szabo, 2015).

Letting b_{ij} be the number of paths of length 2 from $i - j$. Assume that a graph of three vertices exist with $i = 1, k = 2$ and $j = 3$ we think about the quantity $a_{12} * a_{23}$ will yield the possibilities below (Matawa, 2000)

The possibilities are that:

$$a_{12} = 1 \text{ and } a_{23} = 1, \text{ so } a_{12} * a_{23} = 1 \qquad (6.0)$$

$$a_{12} = 0 \text{ and } a_{23} = 1, \text{ so } a_{12} * a_{23} = 0 \qquad (7.0)$$

$$a_{12} = 1 \text{ and } a_{23} = 0, \text{ so } a_{12} * a_{23} = 0 \qquad (8.0)$$

$$a_{12} = 0 \text{ and } a_{23} = 0, \text{ so } a_{12} * a_{23} = 0 \qquad (9.0)$$

Therefore, we have that, $a_{12} * a_{23}$ =1 if there exist a path from vertex 1 to vertex 3 via vertex 2 and 0, otherwise. In general, consider any three vertices i, j and k, let $a_{ik} = 1$if there exist an edge from i to k, otherwise $a_{ik} = 0$, of course, the same can go for a_{ik}. By multiplying a_{ik} and a_{jk} we get $a_{ik} * a_{jk} = 0$ or 1. It will be 1 if both a_{ik} and a_{kj} are 1, otherwise it will be 0. So, $a_{ik} * a_{jk} = 1$ if and only if there exist a path of length 2 from vertex i to j via k and $a_{ik} * a_{jk} = 0$ otherwise.

Since in an adjacency matrix, the diagonal elements are $0's$, a_{ii} and a_{jj} are $0's$ so when $a_{ik} * a_{jk} = 1$, the vertex k is neither i nor j and the sum of:

$$b_{ij} = a_{i1} * a_{1j} + a_{i2} * a_{2j} + a_{i3} * a_{3j} + \ldots + a_{1n} * a_{nj} \tag{10.0}$$

This is so because $a_{i1} * a_{1j}$ contributes a 1, if there exists a path of length 2 from $1\ to\ j$ to via 1(and 0, otherwise). $a_{i2} * a_{2j}$ contributes a 1, if there exists a path of length 2 from i to j via 2 (and 0, otherwise) and $a_{in} * a_{nj}$ contributes a 1, if there exists a path of length 2 from i to j via n (and 0, otherwise). Therefore, b_{ij} is the sum of the number of paths of length 2 from i to j, implying that (10.0) holds.

In other words, according to Stevanović (2015), the adjacency matrix of $G = (V, E)$is the $n \times n$ matrix A indexed by V, whose $(u, v) - entry$ is defined as:

$$A_{UV} = \begin{cases} 1 \\ 0 \end{cases}, \ 1\ if\ uv \in E, undefined\ and\ 0\ if\ uv \notin E \tag{11.0}$$

The number of walks of length $k, k \geq 0$, between vertices u and v in G is equal to $(A^k)^1_{u,v}$.

We will prove this by induction, for the base case when k=0, the unit matrix A^0=1 has entries of 1 for the diagonal and 0 for the non-diagonal entries. These corresponds to the number of walks of length 0, which consist of a single vertex only.

Now, let's assume that the inductive hypothesis holds for some $k \geq 0$. Any walk of length k between vertices u and v, can be represented as an edge connecting uz for some neighbor $Z \in N_u$ and a walk of length $k - 1$ between z and v. According to our inductive hypothesis, the number of walks of length k between u and v is equal to:

$$\sum_{z \in N_u} (A^{k-1})_{z,v} = \sum_{z \in v} A_{u,z} (A^{k-1})_{z,v} = (A^k)_{u,v} \tag{12.0}$$

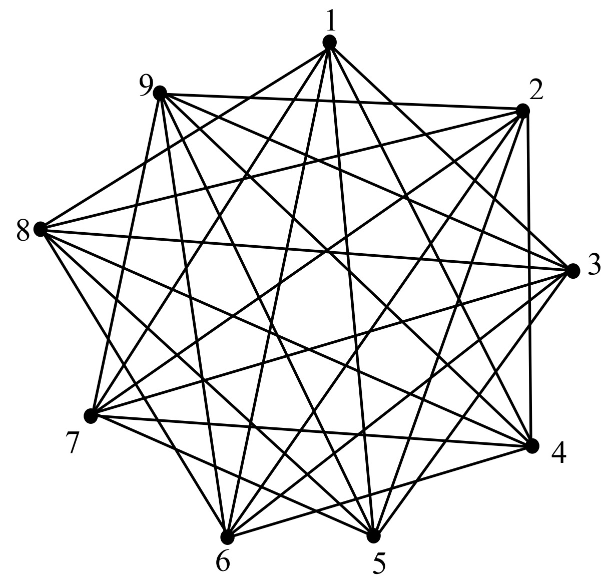
4.0 RESULT

In this section, we focus on presenting the results derived from the techniques employed in our methods. Specifically, for the design construction, we utilized the cyclic method as described in Nguyen (1994), implementing it through JAVA code. Additional seventeen (17) incomplete block designs (IBDs) of the same dimension were constructed as shown in Appendix 2. We subsequently labeled the newly constructed designs as designs G through W in a sequential order. Cosequently, along with the six (6) existing designs proposed by Nguyen (1994), we now have a total set of twenty-three (23) IBDs to profile them for "robustness" in terms of their argencency matric for field experimentation. This work documents a new contribution to the existing literature on the topic of adajacency matrix of Incomplete block designs.

Fig. 1.0 Connected Adjacency Matrix and Concurrence Graphs for Design H

	1	2	3	4	5	6	7	8	9
1	0	1	1	1	0	0	1	1	1
2	1	0	1	1	1	0	0	1	1
3	1	1	0	1	1	1	0	0	1
4	1	1	1	0	1	1	1	0	0
5	0	1	1	1	0	1	1	1	0
6	0	0	1	1	1	0	1	1	1
7	1	0	0	1	1	1	0	1	1
8	1	1	0	0	1	1	1	0	1
9	1	1	1	0	0	1	1	1	0

Adjacency Matrix for Design H

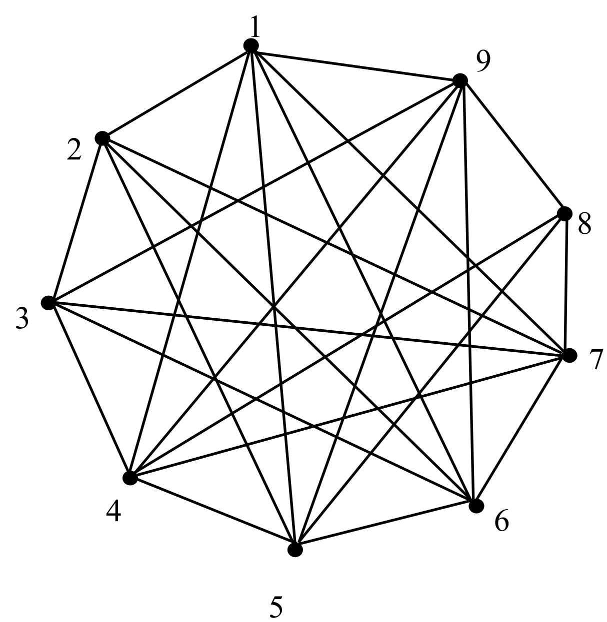


Concurrence Graph "Design H"

Fig. 2.0: Connected Adjacency Matrix and Concurrence Graphs for Design J

	1	2	3	4	5	6	7	8	9
1	0	1	0	1	1	1	1	0	1
2	1	0	1	0	1	1	1	1	0
3	0	1	0	1	0	1	1	1	1
4	1	0	1	0	1	0	1	1	1
5	1	1	0	1	0	1	0	1	1
6	1	1	1	0	1	0	1	0	1
7	1	1	1	1	0	1	0	1	0
8	0	1	1	1	1	0	1	0	1
9	1	0	1	1	1	1	0	1	0

Adjacency Matrix for Design J

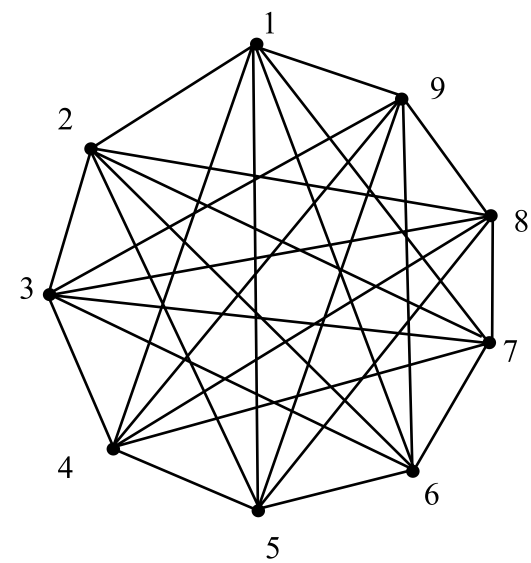


Concurrence Graph "Design J"

Fig. 3.0: Connected Adjacency Matrix and Concurrence Graphs for Design N

	1	2	3	4	5	6	7	8	9
1	0	1	0	1	1	1	1	0	1
2	1	0	1	0	1	1	1	1	0
3	0	1	0	1	0	1	1	1	1
4	1	0	1	0	1	0	1	1	1
5	1	1	0	1	0	1	0	1	1
6	1	1	1	0	1	0	1	0	1
7	1	1	1	1	0	1	0	1	0
8	0	1	1	1	1	0	1	0	1
9	1	0	1	1	1	1	0	1	0

Adjacency Matrix for Design N

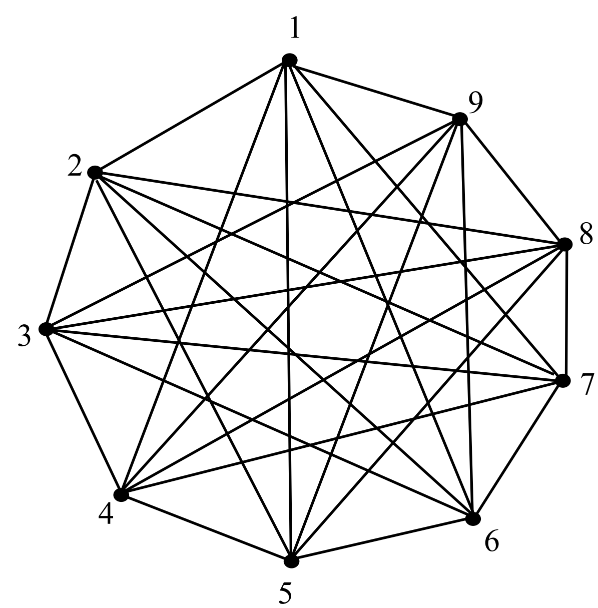


Concurrence Graph "Design N"

4.0: Connected Adjacency Matrix and Concurrence Graphs for Design O

	1	2	3	4	5	6	7	8	9
1	0	1	0	1	1	1	1	0	1
2	1	0	1	0	1	1	1	1	0
3	0	1	0	1	0	1	1	1	1
4	1	0	1	0	1	0	1	1	1
5	1	1	0	1	0	1	0	1	1
6	1	1	1	0	1	0	1	0	1
7	1	1	1	1	0	1	0	1	0
8	0	1	1	1	1	0	1	0	1
9	1	0	1	1	1	1	0	1	0

Adjacency Matrix for Design O

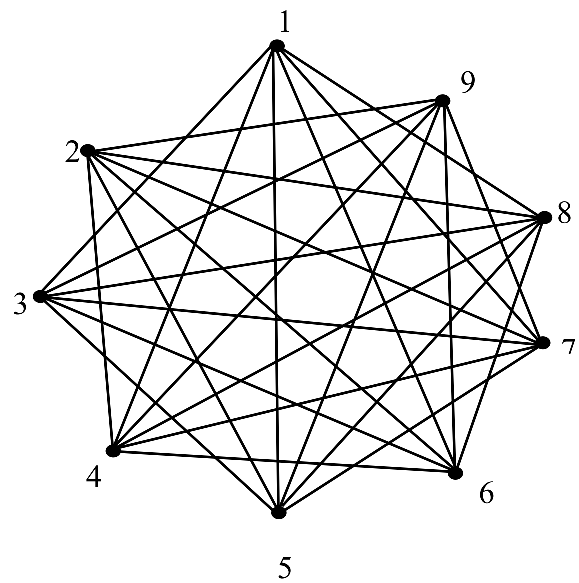


Concurrence Graph "Design O"

Fig. 5. 0: Connected Adjacency Matrix and Concurrence Graphs for Design P

	1	2	3	4	5	6	7	8	9
1	O	O	1	1	1	1	1	1	O
2	O	O	O	1	1	1	1	1	1
3	1	O	O	O	1	1	1	1	1
4	1	1	O	O	O	1	1	1	1
5	1	1	1	O	O	O	1	1	1
6	1	1	1	1	O	O	O	1	1
7	1	1	1	1	1	O	O	O	1
8	1	1	1	1	1	1	O	O	O
9	O	1	1	1	1	1	1	O	O

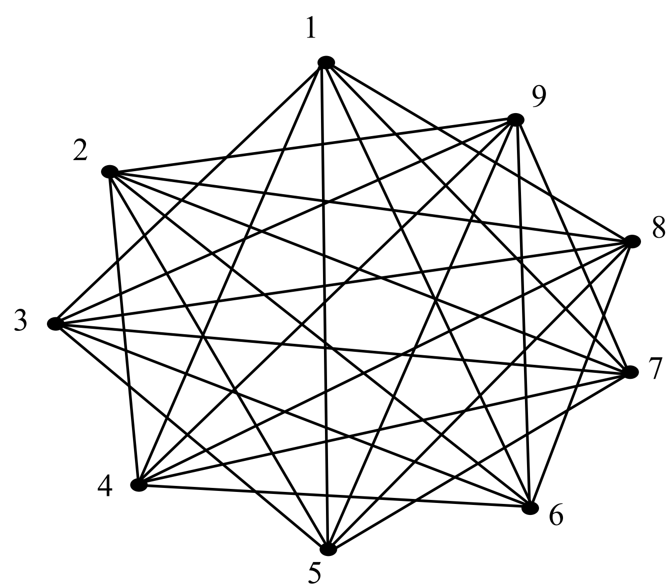
Adjacency Matrix for Design P



Concurrence Graph "Design P"

Fig.6.0: Connected Adjacency Matrix and Concurrence Graphs for Design Q

	1	2	3	4	5	6	7	8	9
1	0	0	1	1	1	1	1	1	0
2	0	0	0	1	1	1	1	1	1
3	1	0	0	0	1	1	1	1	1
4	1	1	0	0	0	1	1	1	1
5	1	1	1	0	0	0	1	1	1
6	1	1	1	1	0	0	0	1	1
7	1	1	1	1	1	0	0	0	1
8	1	1	1	1	1	1	0	0	0
9	0	1	1	1	1	1	1	0	0



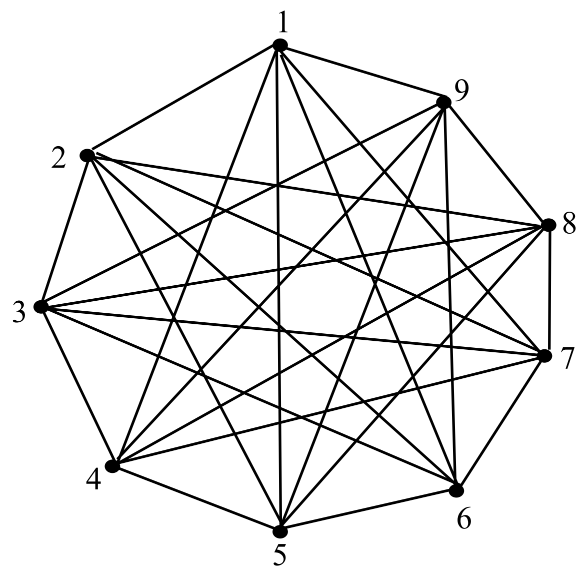
Concurrence Graph "Design Q"

Adjacency Matrix for Design Q

Fig. 7.0: Connected Adjacency Matrix and Concurrence Graphs for Design W

	1	2	3	4	5	6	7	8	9
1	0	1	0	1	1	1	1	0	1
2	1	0	1	0	1	1	1	1	0
3	0	1	0	1	0	1	1	1	1
4	1	0	1	0	1	0	1	1	1
5	1	1	0	1	0	1	0	1	1
6	1	1	1	0	1	0	1	0	1
7	1	1	1	1	0	1	0	1	0
8	0	1	1	1	1	0	1	0	1
9	1	0	1	1	1	1	0	1	0

Adjacency Matrix for Design W



Concurrence Graph "Design W"

Table 1.0: Result Based on Adjacency Matrix (AM) of the Designs

Designs	Numbers of "0's **(Not-Connected)**	Number of "1's" **(Connected)**
A	41	40
B	33	48
C	32	49
D	30	51
E	28	53
F	29	52
G	23	48

H	**27**	**54**
I	45	36
J	**27**	**54**
K	29	52
L	45	36
M	45	36
N	**27**	**54**
O	**27**	**54**
P	27	**54**
Q	**27**	**54**
R	45	36
S	28	53
T	30	51
U	29	52
V	29	52
W	**27**	**54**

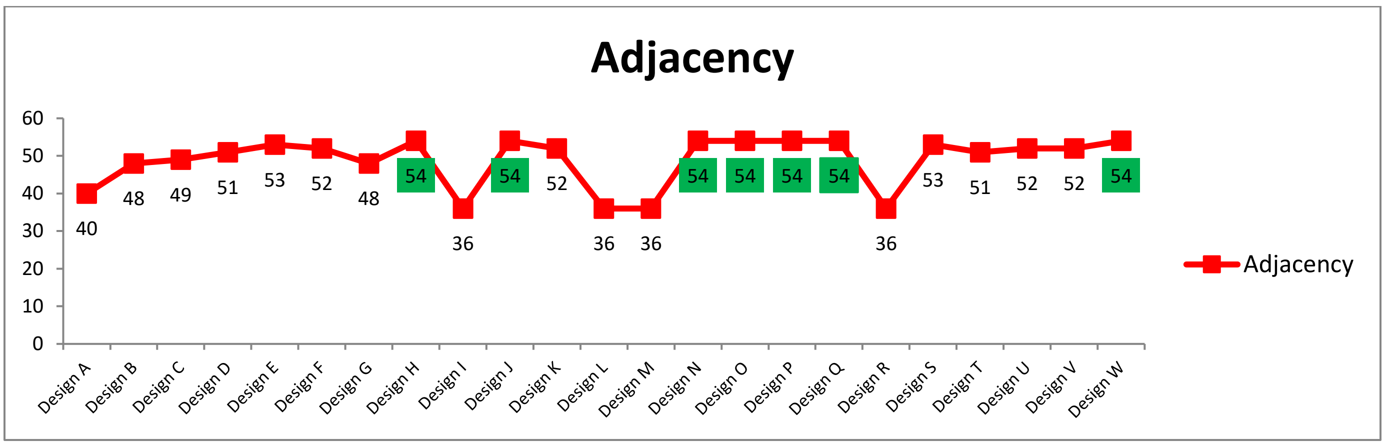


Fig. 8.0: Graph Showing the Best of the Twenty-Three IBDs Based on Adjacency Matrix

5.0 DISCUSSION

The results, as provided in Table 1.0 and Figure 8.0 shows that only design **E** from Nguyen's designs **(A to F)** is connected based on adjacency of treatment combinations in blocks while designs **H, J, N, O, P, Q** and **W** from the newly constructed IBDs are connected based on adjacency of treatment combinations in blocks with 54 adjacent connections. This is supported by the adjacency matrices and concurrence graphs in Figures 1-7. Closely competing with these designs for "bestness" are Nguyen's design E and the newly constructed design S, respectively with 53 adjacent connections, designs U, V, and K with 52 adjacent connections and designs D and T with 51 adjacent connections. The least connected designs based on adjacency of treatment combinations in blocks are designs I, L, M, and R with 36 adjacent connections. The findings in this study regarding the adjacency matrix of the design were reached using the well-known

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properties of an adjacency matrix in Yilmaz and Durmas (2012) which is corroborated by the assertion in Giampiero et al (2017) and the results of Chigbu and Ekpo (2010) which were based on the numbers of "1's" and "0's" in the adjacency matrix of the design.

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Appendix 1: Existing Incomplete Block Designs Due to Nguyen (1994)

3	1	7
9	8	4
5	2	6
3	2	7
5	9	8
6	1	4
5	9	4
2	8	3
7	1	6

(*A*)

3	1	9
7	8	4
5	2	6
3	2	7
5	9	8
6	1	4
5	9	4
2	8	3
7	1	6

(*B*)

3	1	9
7	8	4
5	2	6
9	2	7
5	3	8
6	1	4
5	9	4
2	8	3
7	1	6

(*C*)

3	1	9
7	8	4
5	2	6
3	2	7
5	3	8
6	1	4
5	9	4
2	8	9
7	1	6

(*D*)

3	1	9
7	8	4
5	2	6
1	2	7
5	3	8
6	1	4
5	9	4
2	8	9
7	3	6

(*E*)

3	1	9
7	8	4
5	2	6
9	2	7
5	3	8
6	1	4
5	9	4
2	8	1
7	3	6

(*F*)

Note: (A) - (F) signifies name of designs, e.g., (A) is Design A

Appendix 2: Newly Constructed Set of Seventeen Incomplete Block Designs Using all Nguyen (1994) blocks as initial Blocks.

$$
\begin{bmatrix} 3 & 1 & 7 \\ 4 & 2 & 8 \\ 5 & 3 & 9 \\ 6 & 4 & 1 \\ 7 & 5 & 2 \\ 8 & 6 & 3 \\ 9 & 7 & 4 \\ 1 & 8 & 5 \\ 2 & 9 & 6 \\ & (G) & \end{bmatrix}
\begin{bmatrix} 3 & 1 & 9 \\ 4 & 2 & 1 \\ 5 & 3 & 2 \\ 6 & 4 & 3 \\ 7 & 5 & 4 \\ 8 & 6 & 5 \\ 9 & 7 & 6 \\ 1 & 8 & 7 \\ 2 & 9 & 8 \\ & (H) & \end{bmatrix}
\begin{bmatrix} 9 & 8 & 4 \\ 1 & 9 & 5 \\ 2 & 1 & 6 \\ 3 & 2 & 7 \\ 4 & 3 & 8 \\ 5 & 4 & 9 \\ 6 & 5 & 1 \\ 7 & 6 & 2 \\ 8 & 7 & 3 \\ & (I) & \end{bmatrix}
\begin{bmatrix} 7 & 8 & 4 \\ 8 & 9 & 5 \\ 9 & 1 & 6 \\ 1 & 2 & 7 \\ 2 & 3 & 8 \\ 3 & 4 & 9 \\ 4 & 5 & 1 \\ 5 & 6 & 2 \\ 6 & 7 & 3 \\ & (J) & \end{bmatrix}
\begin{bmatrix} 5 & 2 & 6 \\ 6 & 3 & 7 \\ 7 & 4 & 8 \\ 8 & 5 & 9 \\ 9 & 6 & 1 \\ 1 & 7 & 2 \\ 2 & 8 & 3 \\ 3 & 9 & 4 \\ 4 & 1 & 5 \\ & (K) & \end{bmatrix}
\begin{bmatrix} 3 & 2 & 7 \\ 4 & 3 & 8 \\ 5 & 4 & 9 \\ 6 & 5 & 1 \\ 7 & 6 & 2 \\ 8 & 7 & 3 \\ 9 & 8 & 4 \\ 1 & 9 & 5 \\ 2 & 1 & 6 \\ & (L) & \end{bmatrix}
$$

$$
\begin{bmatrix} 9 & 2 & 7 \\ 1 & 3 & 8 \\ 2 & 4 & 9 \\ 3 & 5 & 1 \\ 4 & 6 & 2 \\ 5 & 7 & 3 \\ 6 & 8 & 4 \\ 7 & 9 & 5 \\ 8 & 1 & 6 \\ & (M) & \end{bmatrix}
\begin{bmatrix} 1 & 2 & 7 \\ 2 & 3 & 8 \\ 3 & 4 & 9 \\ 4 & 5 & 1 \\ 5 & 6 & 2 \\ 6 & 7 & 3 \\ 7 & 8 & 4 \\ 8 & 9 & 5 \\ 9 & 1 & 6 \\ & (N) & \end{bmatrix}
\begin{bmatrix} 5 & 9 & 8 \\ 6 & 1 & 9 \\ 7 & 2 & 1 \\ 8 & 3 & 2 \\ 9 & 4 & 3 \\ 1 & 5 & 4 \\ 2 & 6 & 5 \\ 3 & 7 & 6 \\ 4 & 8 & 7 \\ & (O) & \end{bmatrix}
\begin{bmatrix} 5 & 3 & 8 \\ 6 & 4 & 9 \\ 7 & 5 & 1 \\ 8 & 6 & 2 \\ 9 & 7 & 3 \\ 1 & 8 & 4 \\ 2 & 9 & 5 \\ 3 & 1 & 6 \\ 4 & 2 & 7 \\ & (P) & \end{bmatrix}
\begin{bmatrix} 6 & 1 & 4 \\ 7 & 2 & 5 \\ 8 & 3 & 6 \\ 9 & 4 & 7 \\ 1 & 5 & 8 \\ 2 & 6 & 9 \\ 3 & 7 & 1 \\ 4 & 8 & 2 \\ 5 & 9 & 3 \\ & (Q) & \end{bmatrix}
\begin{bmatrix} 5 & 9 & 4 \\ 6 & 1 & 5 \\ 7 & 2 & 6 \\ 8 & 3 & 7 \\ 9 & 4 & 8 \\ 1 & 5 & 9 \\ 2 & 6 & 1 \\ 3 & 7 & 2 \\ 4 & 8 & 3 \\ & (R) & \end{bmatrix}
$$

$$
\begin{bmatrix} 2 & 8 & 3 \\ 3 & 9 & 4 \\ 4 & 1 & 5 \\ 5 & 2 & 6 \\ 6 & 3 & 7 \\ 7 & 4 & 8 \\ 8 & 5 & 9 \\ 9 & 6 & 1 \\ 1 & 7 & 2 \\ & (S) & \end{bmatrix}
\begin{bmatrix} 2 & 8 & 9 \\ 3 & 9 & 1 \\ 4 & 1 & 2 \\ 5 & 2 & 3 \\ 6 & 3 & 4 \\ 7 & 4 & 5 \\ 8 & 5 & 6 \\ 9 & 6 & 7 \\ 1 & 7 & 8 \\ & (T) & \end{bmatrix}
\begin{bmatrix} 2 & 8 & 1 \\ 3 & 9 & 2 \\ 4 & 1 & 3 \\ 5 & 2 & 4 \\ 6 & 3 & 5 \\ 7 & 4 & 6 \\ 8 & 5 & 7 \\ 9 & 6 & 8 \\ 1 & 7 & 9 \\ & (U) & \end{bmatrix}
\begin{bmatrix} 7 & 1 & 6 \\ 8 & 2 & 7 \\ 9 & 3 & 8 \\ 1 & 4 & 9 \\ 2 & 5 & 1 \\ 3 & 6 & 2 \\ 4 & 7 & 3 \\ 5 & 8 & 4 \\ 6 & 9 & 5 \\ & (V) & \end{bmatrix}
\begin{bmatrix} 7 & 3 & 6 \\ 8 & 4 & 7 \\ 9 & 5 & 8 \\ 1 & 6 & 9 \\ 2 & 7 & 1 \\ 3 & 8 & 2 \\ 4 & 9 & 3 \\ 5 & 1 & 4 \\ 6 & 2 & 5 \\ & (W) & \end{bmatrix}
$$

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Note: (G) - (W) signifies name of the designs, e.g., (G) is Design G, (H) is Design H etc

Appendix 3: Java Output of the Newly Constructed IBDs with Nguyen's Initial Blocks

BLOCK G	BLOCK H	BLOCK I	BLOCK J	BLOCK K	BLOCK L	BLOCK M	BLOCK N	BLOCK O	BLOCK P	BLOCK Q	BLOCK R	BLOCK S	BLOCK T	BLOCK U	BLOCK V	BLOCK W
317	319	984	784	526	327	927	127	598	538	614	594	283	289	281	716	736
428	421	195	895	637	438	138	238	619	649	725	615	394	391	392	827	847
539	532	216	916	748	549	249	349	721	751	836	726	415	412	413	938	958
641	643	327	127	859	651	351	451	832	862	947	837	526	523	524	149	169
752	754	438	238	961	762	462	562	943	973	158	948	637	634	635	251	271
863	865	549	349	172	873	573	673	154	184	269	159	748	745	746	362	382
974	976	651	451	283	984	684	784	265	295	371	261	859	856	857	473	493
185	187	762	562	394	195	795	895	376	316	482	372	961	967	968	584	514
296	298	873	673	415	216	816	916	487	427	593	483	172	178	179	695	625

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