THE WEIBULL INVERSE POWER LOMAX DISTRIBUTION WITH ITS APPLICATIONS TO RELIABILITY DATA

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Abstract

By compounding the Weibull-g family with the Inverse Power Lomax distribution, we were able to create a useful lifetime model known as the Weibull Inverse Power Lomax distribution (WIPLD). Its statistical features were derived using the mixture representation. Properties, namely, quantile, density, reliability, and hazard functions were identified as statistical features. Additional metrics that are determined include the mean, median, moments, incomplete moments, characteristic function, Bonferroni curve, Lorenz curve, stress-strength reliability, Rényi entropy, moment generating function, order statistics, and Tsallis entropy. To estimate the model's parameters, the maximum likelihood technique is applied. WIPLD is superior in terms of application and usability in modeling real-life data, as demonstrated by using two lifetime data sets. The study's information criteria are used to determine the model's goodness-of-fit, and the findings indicate that the model provides the best fit for the actual data sets.

Keywords: Weibull Inverse Power Lomax distribution; incomplete moments; Tsallis entropy; Moments.

1.0 Introduction

In many scientific and technological domains, statistical modeling of lifetime phenomena is an essential component of statistical work. When considering the modeling of lifetime data, standard distributions have been redefined to obtain a new generalization; different transformation/generalization methods have been developed. In order to facilitate effective applications, the transformation typically leads to the addition of one or more shape parameters to the baseline distribution or standard distribution. Over the past century, numerous noteworthy distributions have been created that are used as statistical models in the engineering and

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scientific field. The chief among them is beta-g distribution developed by [12]. Others include: Marshall-Olkin generalized family of distribution by [10]. Beta-gamma-generated distributions studied by [17]. A new family of generalized distributions was proposed by [4], [5] developed the exponentiated-g family. The properties of the Weibull-g were studied by [2], logistic-X family by [15]. The odd Chen family was proposed by [6]. In this study, we focus our aim on modifying the inverse power Lomax distribution ($IPLD$) introduced by [20]. This distribution will serve as the baseline to generate a flexible modification of the $IPLD$ called the $WIPLD$, which is flexible and can be applied in modeling failure data.

Lomax distribution (LD) has been used to model lifetime data especially in applied sciences due to its simplicity and tail. It serve as an alternative model to the Rayleigh, Weibull, Gompertz etc., for further study on the properties of LD, see [3]. This study extends $IPLD$ by using the Weibull-g developed and studied by [1] meant to enhance the scope of applications $IPLD$ in real life application. The new distribution formed by compounding the Weibull-g family and the $IPLD$ called the $WIPLD$, which is flexible with non-monotonic failure rate and can be applied in modeling lifetime data with increasing, decreasing, non-monotonic bathtub failure rate. A random variable X follows an $IPLD$, if its distribution and survival function is respectively, given by

$$\bar{J}(x; \tilde{a}, b, \rho) = \left(1 + \frac{x^{-\tilde{a}}}{b}\right)^{-\rho}, \; x > 0 \tag{1}$$

and

$$J(x; \tilde{a}, b, \rho) = 1 - \left(1 + \frac{x^{-\tilde{a}}}{b}\right)^{-\rho}, \; x > 0; \; \tilde{a}, b, \rho > 0 \tag{2}$$

The associated PDF to (3) is given by

$$j(x; \tilde{a}, b, \rho) = \frac{a\rho}{b} x^{-\tilde{a}-1} \left(1 + \frac{x^{-\tilde{a}}}{b}\right)^{-\rho}. \; x > 0; \; \tilde{a}, b, \rho > 0 \tag{3}$$

The added shape parameter a enables the flexibility of the PDF of the $IPLD$ in such that the PDF of $IPLD$ decreases if $\tilde{a} \leq 1$ and increases if $\tilde{a} > 1$. Using the Weibull-g (W-g) developed by [2]. The CDF of the W-g is presented as follows,

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$$F(x;\text{ã},\beta,\zeta)=\int_{0}^{\frac{J(x;\varsigma)}{\bar{J}(x;\varsigma)}}\text{ã}\beta t^{\beta-1}e^{-\text{ã}t^{\beta}}dt=1-e^{-\theta\left[\frac{J(x,\varsigma)}{\bar{J}(x;\varsigma)}\right]^{\rho}},\qquad(4)$$

Where $J(x,\varsigma)$ represent the CDF of the baseline with parameter vector ζ and $\bar{J}(x;\varsigma)=1-J(x,\varsigma)$. Based on W-G family, in particular, several distributions have been developed by many authors which exist in literature.

2.0 Weibull Inverse Power Lomax distribution

By putting (2) into (4), we obtain a random variable X which follows the $WIPLD$ and the CDF is written as

$$F_{WIPLD}(x;\text{ã},b,\rho,v)=1-e^{\left\{-\theta\left[\left(1+\frac{x^{-\text{ã}}}{b}\right)^{-\rho}-1\right]^{-v}\right\}},\quad x>0;\ \text{ã},b,\rho,v>0\qquad(5)$$

Since, $\frac{\left(1+\frac{x^{-\text{ã}}}{b}\right)^{-\rho}}{1-\left(1+\frac{x^{-\text{ã}}}{b}\right)^{-\rho}}=\left[\left(1+\frac{x^{-\text{ã}}}{b}\right)^{-\rho}-1\right]^{-1}$.

The associated PDF to (5) is written as

$$f_{WIPLD}(x,\text{ã},b,\rho,v)$$
$$=\frac{\text{ã}v\rho\theta}{b}x^{-\text{ã}-1}\left(1+\frac{x^{-\text{ã}}}{b}\right)^{-(\rho v+1)}\left[1-\left(1+\frac{x^{-\text{ã}}}{b}\right)^{-\rho}\right]^{-v-1}e^{\left\{-\theta\left[\left(1+\frac{x^{-\text{ã}}}{b}\right)^{\rho}-1\right]^{-v}\right\}},\qquad(6)$$

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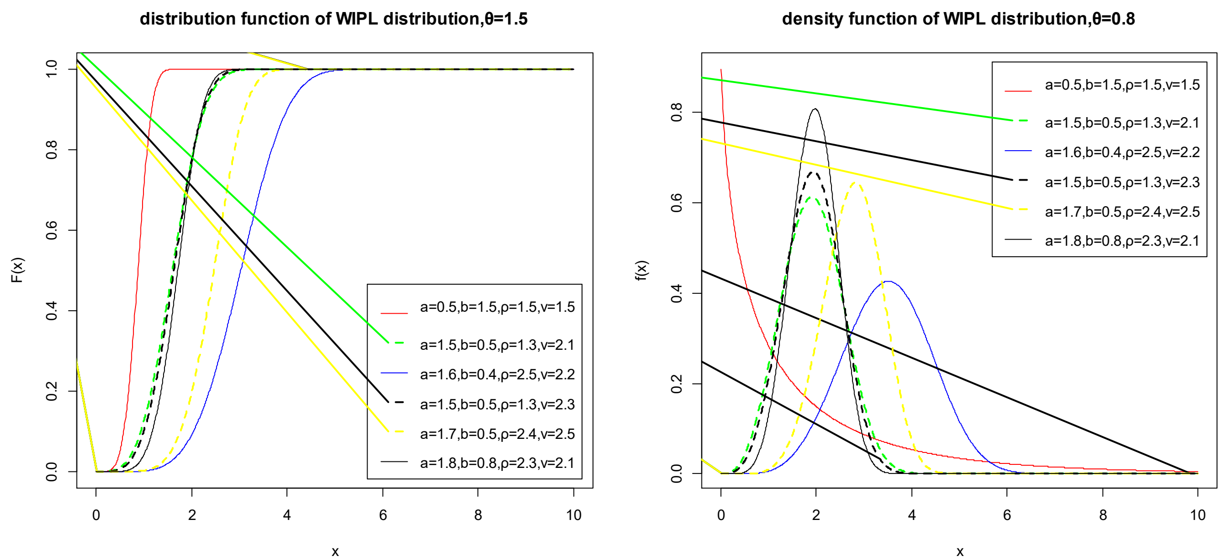

Figure 1.0 CDF and the PDF plots for the $WIPLD$

- Figure 1.0 demonstrates how the PDF of the $WIPLD$ can occasionally be symmetrical, asymmetrical, or reversed J. Additionally, the curves show some degree of adaptability in terms of skewness, mode, and kurtosis, which gives the model some modeling power.

2.1 Reliability properties of the $WIPLD$

Here, we determine a statistical expression for the reliability function of the $WIPLD$, which are of relevance in various real life applications. Taking $\phi = (\tilde{a}, v, \rho, \theta)$, the reliability function, $R_{WIPL}(x;\phi)$, and hazard rate, $h_{WIPL}(x;\phi)$, of the $WIPLD$ are given respectively, as

$$S_{WIPL}(x;\phi) = e^{\left\{-\theta\left[\left(1+\frac{x^{-\tilde{a}}}{b}\right)^{-\rho}-1\right]^{-v}\right\}}, \quad (7)$$

and

$$h_{WIPL}(x;\phi) = \frac{\tilde{a}v\rho\theta}{b}x^{-\tilde{a}-1}\left(1+\frac{x^{-\tilde{a}}}{b}\right)^{-(\rho v+1)}\left[1-\left(1+\frac{x^{-\tilde{a}}}{b}\right)^{-\rho}\right]^{-v-1} \quad (8)$$

the reversed, hrf ($r_{WIPLD}(x;\phi)$) and cumulative hrf ($H_{WIPLD}(x;\phi)$) of X are, respectively, represented as

$$r_{WIPL}\ (x;\phi\ ) = \frac{ãv\rho\theta x^{-ã-1}\left(1+\frac{x^{-ã}}{b}\right)^{-(\rho v+1)}\left[1-\left(1+\frac{x^{-ã}}{b}\right)^{-\rho}\right]^{-v-1} e^{\left\{-\theta\left[\left(1+\frac{x^{-ã}}{b}\right)^{\rho}-1\right]^{-v}\right\}}}{b\left\{1-e^{\left\{-\theta\left[\left(1+\frac{x^{-ã}}{b}\right)^{-\rho}-1\right]^{-v}\right\}}\right\}} \quad (9)$$

and

$$H_{WIPL}\,(x;\phi\ ) = -log[S(x;\zeta)] = -log\left\{-\theta\left[\left(1+\frac{x^{-ã}}{b}\right)^{-\rho}-1\right]^{-v}\right\} \quad (10)$$

and $H(x;\phi) = 0$ for $x \le 0$.

3.0 Important representation

The mixture representation for the density of the $WIPLD$ is derived by applying the series expansion represented as

$$(1-ч)^{-\alpha} = \sum_{p=0}^{\infty}(-1)^{i}\binom{h+p-1}{p}ч^{\alpha} \quad (11)$$

For $|ч| < 1$ and α is a non-negative real non-integer. Applying the series expansion in (11) in (6), the density function of $WIPLD$ becomes

$$f_{WIPL}\,(x;\phi) = \frac{ãv\rho\theta}{b}\sum_{l,m=0}^{\infty}(-1)^{l+m}\frac{\theta^{i}}{i!}\binom{v+l+m}{l}x^{-ã-1}\left(1+\frac{x^{-ã}}{b}\right)^{-[\rho(v+l+m)+1]} \quad (12)$$

3.1 Statistical characteristics of $WIPLD$

3.1.1 Quantiles derivation for the WIPLD

The quantile function (qf) of the $WIPLD$, $Q(q,\phi\ )$, is obtained by solving the non-linear equation: $F(Q(u;\ \phi);\ \phi) = q$ with $0 < q < 1$. Then it follows that

$$x_q = b^{1/ã}\left[\left\{\frac{\left[-\frac{1}{\theta}log(1-q)\right]^{1/v}}{1+\left[-\frac{1}{\theta}log(1-q)\right]^{1/v}}\right\}^{-1/\rho}-1\right]^{-1/ã} \quad (13)$$

The median $(x_{0.5})$ of $WIPLD$ is follows:

$$x_{0.5} = b^{1/\tilde{a}}\left[\left\{\frac{\left[-\frac{1}{\theta}log(0.5)\right]^{1/v}}{1+\left[-\frac{1}{\theta}log(0.5)\right]^{1/v}}\right\}^{-1/\rho} - 1\right]^{-1/\tilde{a}} \quad (14)$$

And the upper quartile is

$$x_{0.75} = b^{1/\tilde{a}}\left[\left\{\frac{\left[-\frac{1}{\theta}log(0.25)\right]^{1/v}}{1+\left[-\frac{1}{\theta}log(0.25)\right]^{1/v}}\right\}^{-1/\rho} - 1\right]^{-1/\tilde{a}} \quad (15)$$

Expression for the coefficient of skewness as presented by [7] and the coefficient of kurtosis by [21] are respectively defined by

$$B = \frac{q_{1/4} - 2q_{1/2} + q_{3/4}}{q_{3/4} - q_{1/4}}$$

and

$$M = \frac{q_{7/8} - q_{5/8} + q_{3/8} - q_{1/8}}{q_{6/8} - q_{2/8}}$$

The sign of B gives the direction of the skewness (skewed left if B is less than zero, almost symmetrical if $B = 0$, and right skewed if B is greater zero).

Table 1.0 present the numerical values for the lower ($q_{1/8}$ median or middle $q_{4/8}$, upper quartiles $q_{6/8}$, B and the M for fixed values of $\tilde{a} = 2.0$, $b = 1.5$ and $\rho = 1.5$ with varying values of θ, and v.

Table 1.0 Skewness and Kurtosis of $WIPLD$

θ, v	$q_{1/8}$	$q_{1/4}$	$q_{3/8}$	$q_{1/2}$	$q_{5/8}$	$q_{6/8}$	$q_{7/8}$	B	M
0.1,0.1	6.3970	295.56	3440.30	24000	136162	768008	5826382	0.9382	7.4164
0.5,0.5	0.5429	0.9906	1.5117	2.1577	3.0015	4.2022	6.2678	0.3732	1.3186
1.0,0.5	0.3293	0.5740	0.8399	1.1586	1.5704	2.1577	3.1753	0.2618	1.3358

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3,1.0	0.4565	0.6068	0.7330	0.8553	0.9855	1.1395	1.3577	0.0671	1.2177
2.5,1.5	0.7051	0.8634	0.7330	1.0999	1.2141	1.3419	1.5124	0.0114	1.2127
4.0,5.0	1.1851	1.2662	1.3215	1.3672	1.4097	1.4535	1.5069	-0.0785	1.2470
6.0,7.5	1.2767	1.3350	1.3738	1.4056	1.4346	1.4643	1.4999	-0.0918	1.2526
8.0,10.0	1.3324	1.3781	1.4082	1.4326	1.4549	1.4774	1.5043	-0.0984	1.2613

- It can be concluded from Table 1.0, that the $WILPD$ is applicable to model lifetime data which are skewed in any direction different shapes of the kurtosis.

3.2. The r^{th} moment

If $X \sim WIPLD(\phi)$, the $\boldsymbol{r^{th}}$ moment of X can be derived using

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f_{WIPLD}(x;\phi)dx. \tag{16}$$

Plugging (12) in (16), we obtain

$$\mu_r' = \frac{\tilde{a}v\rho\theta}{b}\sum_{l,m=0}^{\infty}(-1)^{l+m}\frac{\theta^l}{l!}\binom{v+l+m}{m}\int_{-\infty}^{\infty} x^{r-\tilde{a}-1}\left(1+\frac{x^{-\tilde{a}}}{b}\right)^{-[\boldsymbol{\rho(v+l+m)+1}]} dx \tag{17}$$

Letting, $d = \frac{x^{-\tilde{a}}}{b}$, $dx = -\frac{b^{-1/\tilde{a}}d^{-1/\tilde{a}-1}}{\tilde{a}}$, and further $y = \frac{\text{ʒ}}{1-\text{ʒ}}$, we obtain

$$\mu_r' = v\rho\theta\sum_{l,m=0}^{\infty}(-1)^{l+m}\frac{\theta^l}{l!}\binom{v+l+m}{m}b^{\tilde{a}-r-1}\int_{-\infty}^{\infty} \text{ʒ}^{-r/\tilde{a}}(1-\text{ʒ})^{\frac{r}{\tilde{a}}+\boldsymbol{\rho(v+l+m)-1}}d\text{ʒ} \tag{18}$$

finally, we have

$$\mu_r' = v\rho\theta\sum_{l,m=0}^{\infty}(-1)^{l+m}\frac{\theta^l}{l!}\binom{v+l+m}{m}b^{\tilde{a}-r-1}B\left[(1-r/\tilde{a}),\frac{r}{\tilde{a}} + \rho(v+l+m)\right] \tag{19}$$

Where $B(h,q) = \int_0^1 \text{v}^{h-1}(1-\text{v})^{q-1}$, is the standard beta function $h > 0$, $q > 0$.

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Other numerical values of μ_r' and all other related measures can be obtained from (19). We obtain the mean (μ), the variance (σ^2) and the coefficient of variation (CV) of X estimate using $\mu_1' = \mu$, the $\sigma^2 = \mu_2' - (\mu_1')^2$ and the $CV = {(\mu_2)^{1/2}}/{\mu}$

The mean of $WIPLD$ can be estimated by substituting for $r = 1$ in equation (19). Then we have

$$\mu_1' = v\rho\theta \sum_{l,m=0}^{\infty} (-1)^{l+m} \frac{\theta^l}{l!} \binom{v+l+m}{m} b^{\tilde{a}-2} B\left[\left(1 - {}^{1}/_{\tilde{a}}\right), \frac{1}{\tilde{a}} + \rho(v+l+m)\right] \quad (20)$$

Table 2.0 provides values for first six moments, variance (μ) and the coefficient of variation (CV) with parameters v = 4.1, w = 2.1 and varying the values of c.

Table 2.0. Moments, μ_2 and CV of $WIPLD$

Moments	$c = 3.5$	$c = 1.5$	$c = 0.5$	$c = 8.5$
μ_1'	0.5338	0.5621	1.1451	0.4890
μ_2'	0.3242	0.3668	3.3666	0.2483
μ_3'	0.1820	0.2637	6.6130	0.1299
μ_4'	0.1132	0.2036	22.9181	0.0696
μ_5'	0.0727	0.1664	94.1156	0.0382
μ_6'	0.0480	0.1425	444.7403	0.02132
μ_2	0.0393	0.0508	2.0554	0.0091
CV	0.3714	0.4010	1.2520	0.1962

By extension, the r^{th} incomplete moment of WIPLD is derived as follows;

$$E(X^s) = \int_{-\infty}^{t} f_{WIPLD} x^r (x;\phi) dx \quad (21)$$

Plugging (12) in (21), we obtain

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$$\zeta_r' = \frac{\tilde{a}v\rho\theta}{b}\sum_{l,,m=0}^{\infty}(-1)^{l+m}\frac{\theta^l}{l!}\binom{v+l+m}{m}\int_{-\infty}^{t} x^{r-\tilde{a}-1}\left(1+\frac{x^{-\tilde{a}}}{b}\right)^{-[\boldsymbol{\rho}(\boldsymbol{v}+\boldsymbol{l}+\boldsymbol{m})+\mathbf{1}]} dx \qquad (22)$$

Letting, $m = \frac{x^{-\tilde{a}}}{b},\ dx = -\frac{b^{-1/\tilde{a}}m^{-1/\tilde{a}-1}}{\tilde{a}},$ and further $y = \frac{ʐ}{1-ʐ}$, we obtain

$$\zeta_r' = v\rho\theta\sum_{i,j=0}^{\infty}(-1)^{i+j}\frac{\theta^i}{i!}\binom{v+i+j}{j} b^{\tilde{a}-r-1}\int_{-\infty}^{t} ʐ^{-r/\tilde{a}}(1-ʐ)^{\frac{r}{\tilde{a}}+\boldsymbol{\rho}(\boldsymbol{v}+\boldsymbol{i}+\boldsymbol{j})-\mathbf{1}} dʐ \qquad (23)$$

And finally, we have an expression for the r^{th} incomplete moments of the $WIPLD$ as

$$\zeta_r' = v\rho\theta\sum_{l,m=0}^{\infty}(-1)^{l+m}\frac{\theta^l}{l!}\binom{v+l+m}{m} b^{\tilde{a}-r-1}\, B\left[\frac{t^{-\tilde{a}}}{b};\left(1-{}^{r}/_{\tilde{a}}\right),\frac{r}{\tilde{a}}+\rho(v+i+j)\right] \qquad (24)$$

where $B(q;n,p) = \int_0^q j^{n-1}(1-j)^{p-1}dj$ *, is the beta function.*

3.2 generating (mgf) and Characteristic functions of $WIPLD$

Using the expansion given by $e^{tx} = \int_{z=0}^{\infty} {t^z x^z}/_{z!}$, the mgf of $WIPLD$ are presented as follows:

$$\phi_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f_{WIPLD}(x;\phi)dx \qquad (25)$$

$$= \sum_{r=0}^{\infty} {t^r E(x^r)}/_{r!} = v\rho\theta\sum_{l,m,r=0}^{\infty}(-1)^{l+m}\frac{\theta^l t^r}{r!\,l!}\binom{v+l+m}{m} b^{\tilde{a}-r-1}\, B_{(1-{}^{r}/_{\tilde{a}})}\left[\frac{r}{\tilde{a}}+\rho(v+l+m)\right]$$

We may also determine the characteristic function of the based on the r^{th} moments of the $WIPLD$ as

$$\phi_X(t) = E\left(e^{itx}\right) = \int_{-\infty}^{\infty} e^{itx} f(x)dx = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(X^r) \qquad (26)$$

$$= v\rho\theta\sum_{l,m,r=0}^{\infty}(-1)^{l+m}\frac{\theta^i(it)^r}{r!\,l!}\binom{v+l+m}{m} b^{\tilde{a}-r-1}\, B_{(1-{}^{r}/_{\tilde{a}})}\left[\frac{r}{\tilde{a}}+\rho(v+l+m)\right]$$

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3.3 Mean deviation

The degree of dispersion in a population is assessed using the mean deviation, which is similar to the median. Suppose $\mu=\bar{X}$ and M represents the mean and the median of the $WIPLD$ given in (20) and (14) respectively. The mean deviation of $WIPLD$ about the mean is obtained as follows:

$$\Gamma_1(X) = E|X-\mu| = \int_0^\infty |X-\mu| f(x;\phi)dx, \tag{27}$$

$$= 2\mu F(\mu; v, w, \lambda) - 2\mu + 2\int_\mu^\infty x f(x;\phi)dx$$

$$= 2\mu\left\{1 - e^{\left\{-\theta\left[\left(\mathbf{1}+\frac{\boldsymbol{x}^{-\tilde{a}}}{\boldsymbol{b}}\right)^{-\rho}-1\right]^{-v}\right\}}\right\} - 2\mu + 2v\rho\theta\sum_{i,j=0}^{\infty}(-1)^{i+j}\frac{\theta^i}{i!}\binom{v+i+j}{j}b^{\tilde{a}-2}$$

$$\times B\left[\frac{\mu^{-\tilde{a}}}{b}; \left(1-{}^{1}/_{\tilde{a}}\right), \frac{1}{\tilde{a}} + \rho(v+i+j)\right] \tag{28}$$

Mean deviation about the median is derived as

$$\Gamma_2(X) = E|X-M| = \int_0^\infty |X-M| f(x;\phi)dx, \tag{29}$$

$$= -\mu + 2\int_m^\infty x f(x;\phi)dx$$

$$= -\boldsymbol{\mu} + 2v\rho\theta\sum_{i,j=0}^{\infty}(-1)^{i+j}\frac{\theta^i}{i!}\binom{v+i+j}{j}b^{\tilde{a}-2}B\left[\frac{m^{-\tilde{a}}}{b}; \left(1-{}^{1}/_{\tilde{a}}\right), \frac{1}{\tilde{a}} + \rho(v+i+j)\right]$$

(30)

3.4 Bonferroni and Lorenz curves

Both the Bonferroni curve and the Lorenz curve are used as a statistical tool in income and poverty analysis, insurance, and reliability analysis among many others. The Bonferroni curve is defined by

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$$B(p) = \frac{1}{p\mu}\int_{0}^{q} xf(x;\phi)dx \tag{30}$$

And Lorenz curve is given by

$$L(p) = \frac{1}{\mu}\int_{0}^{q} xf(x;\phi)dx \tag{31}$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$. In the case of $WIPLD$, we obtain

$$B(p) = \frac{v\rho\theta}{p\mu}\sum_{i,j=0}^{\infty}(-1)^{l+m}\frac{\theta^{i}}{i!}\binom{v+l+m}{m} b^{\tilde{a}-2}\, B\left[\frac{p^{-\tilde{a}}}{b};\left(1-{}^{1}/_{\tilde{a}}\right),\frac{1}{\tilde{a}}+\rho(v+l+m)\right]$$

and

$$L(p) = \frac{v\rho\theta}{\mu}\sum_{i,j=0}^{\infty}(-1)^{l+m}\frac{\theta^{l}}{l!}\binom{v+l+m}{j} b^{\tilde{a}-2}\, B\left[\frac{q^{-\tilde{a}}}{b};\left(1-{}^{1}/_{\tilde{a}}\right),\frac{1}{\tilde{a}} + \rho(v+l+m)\right] \tag{32}$$

3.5 Stress strength Reliability for $WIPLD$

Suppose Y and Z represents an independent random variable which follows the $WIPLD$ containing sets of the following parameters $(\tilde{a}, b, \rho_1, v_1, \theta_1)$ and$(\tilde{a}, b, \rho_2, v_2, \theta_2)$, respectively. A useful expression for the stress-strength reliability is given by $R = P(X_2 < X_1)$.

$$R = P(X_1 < X_2 = \int_{-\infty}^{\infty} f_1(\tilde{a}, b, \rho_1, v_1, \theta_1)F_2(\tilde{a}, b, \rho_2, v_2, \theta_2)\, dx, \tag{33}$$

Consequently, we can write

$$R = F_1(a, b, \rho_1, v_1, \theta_1) - \tilde{a}b\rho_1 v_1 \sum_{i=j=p=q}^{\infty}(-1)^{i+j}\theta_1^{j+1}\theta_2^{i}\binom{v_1(j+1)+k}{k}\binom{v_2 i+q-1}{l}$$

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$$\times \int_{0}^{\infty} x^{-b-1} \left(1+\frac{x^{-b}}{b}\right)^{-[\rho_1 v_1 (j+1)+\rho_1 p+\rho_2 (i v_2+l+1]} dx \quad (34)$$

$$= F_1(\tilde{a}, b, \rho_1, v_1, \theta_1) - \tilde{a} b \rho_1 v_1 \sum_{i=j=p=q}^{\infty} (-1)^{i+j} \theta_1^{j+1} \theta_2^{i} \binom{v_1(j+1)+p}{p} \binom{v_2 i+q-1}{q}$$

$$\times B(1, [\rho_1 v_1 (j+1)+\rho_1 p+\rho_2 (i v_2+q+1]) \quad (35)$$

4.0 Order Statistics

Suppose $X_{(1),} \ldots, X_{(n)}$ be an ordered sample from the $WIPLD$, the PDF of $X_{(r)}$ is determined by

$$f_r(x;\zeta) = \frac{1}{B(q, n-q+1)} F_{\text{WIPLD}}(x;\phi)^{q-1} [1 - F_{\text{WIPLD}}(x;\phi)]^{n-q} f_{\text{WIPLD}}(x;\phi) \quad (36)$$

Using the series expansion given by

$$(1-b)^q = \sum_{m=0}^{\infty} (-1)^m \binom{q}{m} b^i \quad (37)$$

in equation (32), we have

$$f_q(x;\phi) = \frac{1}{B(q, n-q+1)} \sum_{i=1}^{n-r} (-1)^i \binom{n-q}{i} f_{\text{WIPLD}}(x;\phi) F_{\text{WIPLD}}(x;\phi)^{q+i-1} \quad (38)$$

Now, by substituting equation (5) and (6) in $f_q(x;\phi)$, followed by algebraic manipulation, we derived an expression for the r^{th} order statistics as

$$f_q(x;\phi) = \frac{\frac{\tilde{a} v \rho \theta}{b}}{B(q, n-q+1)} \sum_{i=1}^{n-r} (-1)^i \binom{n-q}{i} x^{-(\tilde{a}+1)} \left(1+\frac{x^{-\tilde{a}}}{b}\right)^{-(\rho v+1)} \left[1 - \left(1+\frac{x^{-\tilde{a}}}{b}\right)^{-\rho}\right]^{-v-1}$$

$$\times e^{\left\{-\theta(q+i)\left[\left(1+\frac{x^{-\tilde{a}}}{b}\right)^{\rho} - 1\right]^{-v}\right\}} \quad (39)$$

5.0 Rényi entropy

The Rényi entropy was developed by [14] and it is given by

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$$I_R^{(\varsigma)} = \frac{1}{1-\varsigma} log[Y_L], \quad \varsigma > 0, \varsigma \neq 1 \tag{40}$$

where

$$Y_L = \int_0^\infty f_{\text{WIPLD}}(x;\phi)^\varsigma \tag{41}$$

Putting equation (6) in (38), we have

$$Y_L = \int_0^\infty \left(\frac{\tilde{a}\nu\rho\theta}{b} x^{-(\tilde{a}+1)} \left(1+\frac{x^{-\tilde{a}}}{b}\right)^{-(\rho\nu+1)} \left[1 - \left(1+\frac{x^{-\tilde{a}}}{b}\right)^{-\rho}\right]^{-\nu-1} e^{\left\{-\theta\left[\left(1+\frac{x^{-\tilde{a}}}{b}\right)^{\rho}-1\right]^{-\nu}\right\}} \right)^\varsigma \tag{42}$$

Solving further, using Taylor series given in (11), we have

$$Y_L = \left(\frac{\tilde{a}\nu\rho\theta}{b}\right)^\varsigma \sum_{i=j=0}^{\infty} \frac{(-1)^i(\varsigma\theta)^i}{i!} \binom{\nu+i+j}{j} \int_{-\infty}^{\infty} x^{-\varsigma(\tilde{a}+1)} \left(1+\frac{x^{-\tilde{a}}}{b}\right)^{-[\rho(j+i+\varsigma\nu)+\varsigma]} dx \tag{43}$$

Letting $y = \frac{x^{-\tilde{a}}}{b}$, plugging it in (43), we derived an expression for Y_L as

$$Y_L = \mathcal{M}_L B\left\{\frac{(\tilde{a}+1)(\varsigma-1)}{\tilde{a}} + 1, [\rho(i+j+\varsigma\nu] - \frac{(\tilde{a}+1)(\varsigma-1)}{\tilde{a}} + 1\right\} \tag{44}$$

where

$$\mathcal{M}_L = \left(\frac{\nu\rho\theta}{b}\right)^\varsigma b^{\varsigma(\tilde{a}+1)} \tilde{a}^{\varsigma-1} \sum_{i=j=0}^{\infty} \frac{(-1)^i(\varsigma\theta)^i}{i!} \binom{\nu+i+j}{j} \tag{45}$$

By putting (44) in (40), we obtain mathematical expression for Rényi entropy of $WIPLD$ as

$$I_R^{(\varsigma)} = \frac{1}{1-\varsigma} log\left[\mathcal{M}_L B\left\{\frac{(\tilde{a}+1)(\varsigma-1)}{\tilde{a}} + 1, [\rho(i+j+\varsigma\nu] - \frac{(\tilde{a}+1)(\varsigma-1)}{\tilde{a}} + 1\right\}\right]$$

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5.2 Tsallis Entropy

The Tsallis entropy was proposed by [8] and later by [16]. The Tsallis entropy of the $WIPLD$ is defined by

$$I_T^{(\Upsilon)} = \frac{1}{\Upsilon - 1}\left[1 - \int_0^{\infty} f_{\text{WIPLD}}(x;\phi)^{\varsigma}\right], \quad \varsigma > 0, \varsigma \neq 1 \tag{46}$$

Since,

$$\int_0^{\infty} f(x;\zeta)^{\Upsilon} = \mathcal{M}_L B\left\{\frac{(\tilde{a}+1)(\varsigma-1)}{\tilde{a}} + 1, [\rho(i+j+\varsigma v] - \frac{(\tilde{a}+1)(\varsigma-1)}{\tilde{a}} + 1\right\}$$

$$I_T^{(\Upsilon)} = \frac{1}{\Upsilon - 1}\left[1 - \left(\mathcal{M}_L B\left\{\frac{(\tilde{a}+1)(\varsigma-1)}{\tilde{a}} + 1, [\rho(i+j+\varsigma v] - \frac{(\tilde{a}+1)(\varsigma-1)}{\tilde{a}} + 1\right\}\right)\right],$$

Where $\mathcal{M}_L$ is defined in (45).

6.0 Maximum likelihood Estimates for the parameters of $WIPLD$

Suppose $\underline{x} = x_1 \dots, x_n$ is set of independent sample from the $WIPLD$. The likelihood function $L(x;\ \zeta)$ corresponding to (43) is given as

$$L(x;\zeta) = \frac{\tilde{a}v\rho\theta}{b}\prod_{i=1}^{n} x^{-(\tilde{a}+1)}\left(1 + \frac{x^{-\tilde{a}}}{b}\right)^{-(\rho v+1)}\left[1 - \left(1 + \frac{x^{-\tilde{a}}}{b}\right)^{-\rho}\right]^{-v-1} e^{\left\{-\theta\left[\left(1+\frac{x^{-\tilde{a}}}{b}\right)^{\rho} - 1\right]^{-v}\right\}} \tag{47}$$

and the log-likelihood function $logL(x;\zeta) = l(x;\zeta)$ is

$$l(x;\zeta) = nlog\left(\frac{\tilde{a}v\rho\theta}{b}\right) - (\tilde{a}+1)\sum_{i=1}^{n} log\,(x_i) - (\rho v + 1)\sum_{i=1}^{n} log\left(1 + \frac{x_i^{-\tilde{a}}}{b}\right)$$

$$-(v+1)\sum_{i=1}^{n}\left[1 - \left(1 + \frac{x_i^{-\tilde{a}}}{b}\right)^{-\rho}\right] - \theta\sum_{i=1}^{n}\left[\left(1 + \frac{x_i^{-\tilde{a}}}{b}\right)^{\rho} - 1\right]^{-v} \tag{48}$$

In literature, the details of the maximum likelihood (ML) method and its procedures can be found.

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6.1 Applications of the $WIPLD$

Here, we make use two lifetime data sets to establish the flexibility of the proposed $WIPLD$. Data I was previously studied by [9], it represent the remission times (in months) of 128 bladder cancer patients. Data II represents the survival times (in days) of 72 guinea pigs reported by [2]. We evaluate WIPLD against a few other competitive lifetime models, such as Weibull Power Lomax (WPL), Lehman type-2 Fréchet Poisson by [11] and the Inverse Power Lomax distributions, using some criteria namely, Akaike information Criterion (AKIC), Bayesian Information Criterion (BAIC), Consistent Akaike Information Criterion (CAKIC), Hannan Quinin Information Criterion (HAQIC), Anderson Darling (D^*), Kolmogorov–Smirnov(K), and Cramér–von Mises (C^*), with its p-Value (P). The model with the best fitted capability will correspond to the model with the smallest AKIC, BAIC, CAKIC, HAQIC, D^*, C^*, K, and the largest P values. Violin plot and the Total Test on Time (TTT) plots are constructed for the two lifetime data sets.

Application I: (Cancer Remission time data)

Violin and TTT plots are given in Figure 2 which proves that the data is skewed to the right and possessing non-monotone failure rate. The MLEs, the standard errors are given in Table 3. Goodness-of-fit statistics like AKIC, BAIC, CAKIC, HAQIC, D^*, K, C^* and P values are displayed in Table 4.

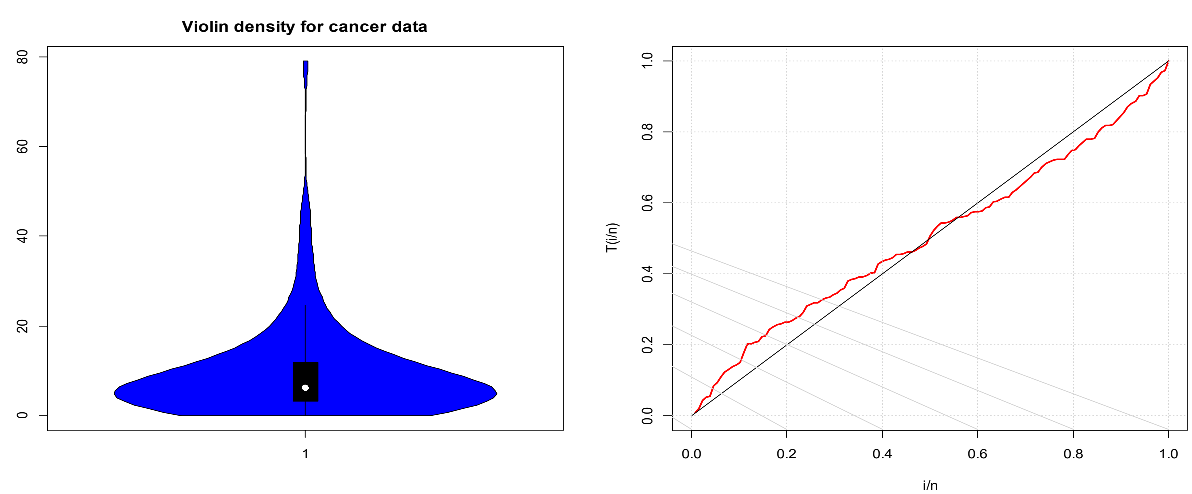

Figure 2. Violin and the TTT plots for data I

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Table 3. MLE's, and Standard error (in parenthesis) for data I

$model$	Ã	b	ρ	θ	v
$WIPLD$	1.3596 (0.6361)	0.0162 (0.0196)	4.7076 (5.3454)	0.3616 (0.2183)	2.9440 (2.4982)
WPL	− (−)	0.1495 (0.0516)	1.9897 (1.0849)	0.7446 (0.2506)	2.3095 (1.3824)
LFP	− (−)	7.0604 (0.5734)	0.2434 (0.0310)	7.0686 (3.2671)	9.1513 (4.1383)
IL	− (−)	0.4986 (0.1569)	2.4604 (0.5922)	− (−)	− (−)

Table 4. goodness-of-fit measurement for data I

$Model$	L	$AKIC$	$BAIC$	$CAKIC$	$HAQIC$	D^*	C^*	K	P
$WIPLD$	409.67	829.33	843.59	829.82	835.13	0.1284	0.0187	0.0346	0.998
WPL	411.11	830.54	841.63	830.22	834.85	0.3152	0.0471	0.0560	0.8162
LFP	412.13	832.25	843.66	832.58	836.89	0.0459	0.4314	0.0363	0.9505
IL	424.68	853.35	859.06	853.45	855.67	1.4643	0.2229	0.1187	0.0544

The WIPLD is best fitted model for the cancer data than the other competing models because the values of it shows the smallest values of AKIC, BAIC, HAQIC, CAKIC, K, D^* and C^*, criteria of goodness of fit for the $WIPL\ D$ with higher P value.

Application II (Pig data)

The violin and TTT plots are given in Figure 3 which shows that the data is skewed to the right and also with non-monotone failure rate. The MLEs, their standard errors (in parentheses) are given in Table 5. Goodness-of-fit statistics like AKIC, BAIC, CAKIC, HAQIC, D^*, C^*, K and P values are displayed in Table 6.

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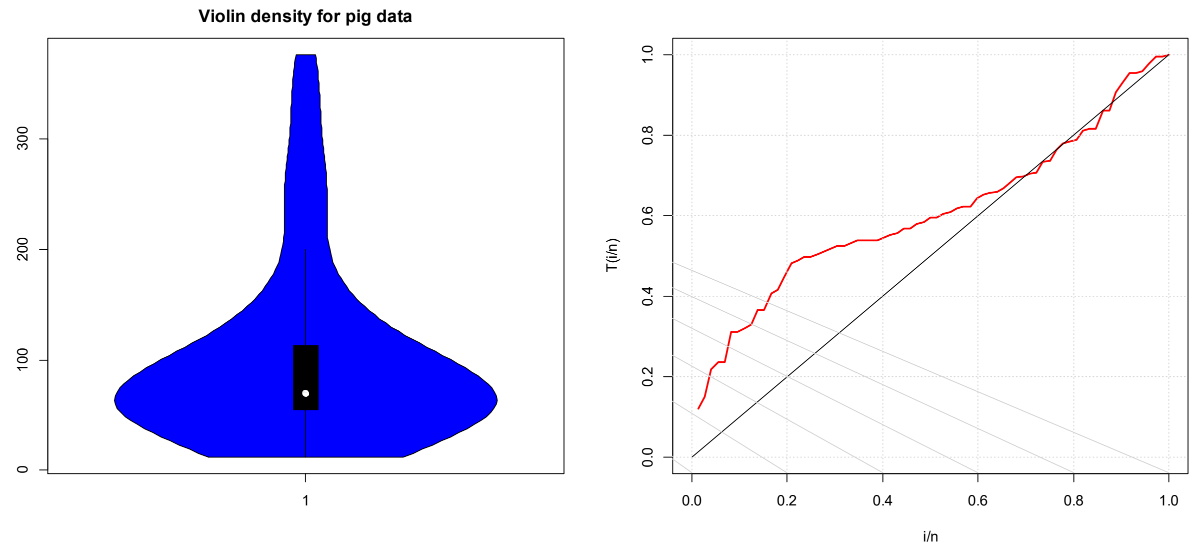

Figure 3. Violin and the TTT plots for the cancer data

Figure 3 indicates that the cancer remission time data is positively skewed

Table 5. MLE's, and Standard error (in parenthesis) for data II

$model$	ã	b	ρ	θ	v
$WIPLD$	1.1332 (0.1941)	0.0133 (0.0095)	4.1797 (2.5831)	0.4842 (0.2176)	8.3658 (8.3658)
WPL	− (−)	0.0486 (0.0394)	2.1984 (1.8407)	0.8749 (0.4096)	6.8437 (2.6331)
LFP	− (−)	10.4198 (1.7851)	0.9314 (0.2445)	−10.6471 (5.2258)	3.8890 (2.7067)
IL	− (−)	0.1957 (0.1236)	12.2834 (7.3055)	− (−)	− (−)

Table 6. Measures of goodness-of-fit for data II

$Model$	l	$AKIC$	$BAIC$	$CAKIC$	$HAQIC$	D^*	C^*	K	P
$WIPLD$	389.66	789.32	800.70	790.23	793.85	0.6554	0.1184	0.0955	0.5275
WIL	390.82	789.65	790.24	790.24	793.27	1.0296	0.1917	0.1144	0.3027

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LFP	390.05	788.11	797.22	788.71	791.733	0.6884	0.1244	0.0987	0.4835
IL	403.85	811.71	816.26	811.88	813.52	0.8015	0.1368	0.1918	0.0099

The WIPLD is best fitted model for the pig data than the other competing models because the values of it shows the smallest values of AKIC, BAIC, HAQIC, CAKIC, K, D^* and C^*, of goodness of fit for the $WIPLD$ with higher P value.

7.0 Conclusion

The distribution and its attributes, including descriptive measures based on quantiles, moments, stress-strength reliability and multi-component stress-strength reliability model, Rényi and Tsallis entropies, have been proposed and developed by us. Estimates of maximum likelihood are calculated. The distribution is the best fit for the two data sets taken into consideration in the study, according to the goodness-of-fit metric. The pig data and cancer remission time data are used to illustrate the applicability of the model.

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