AN EFFICIENT SHORT CUT METHOD FOR COMPUTING THE COEFFICIENTS OF THE BEST LINEAR UNBIASED ESTIMATOR OF POPULATION MEAN FOR CORRELATED RANDOM VARIABLES

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Abstract

An efficient short cut method for computing the coefficients of the best linear unbiased estimator (BLUE) of the population mean for correlated random variables with a defined covariance structure has been proposed in the paper. For correlated random variables with a moving average process of order one covariance structure, the existing method involves minimizing the variance of BLUE subject to the linear constraint that arises from the unbiasedness condition. To propose a new efficient short cut method, the symmetric pattern of BLUE's vector of coefficients or weights was generalized using mathematical induction. The existing quadratic programming problem was further simplified to obtain an efficient short cut computational method by adding the developed symmetric pattern of the vector of coefficients, along with the unbiasedness condition, as constraints. Hence, it reduces the computational time and complexity involved in evaluating the covariance and/or correlation matrix of the correlated variables. The efficacy of the proposed efficient method was demonstrated with ease through the applicability of the method to compute the algebraic expressions of weights or coefficients of BLUE when the number of observations; $n=2k$ and $n=2k+1$ at fixed $k=1,2,...,8$. Then, the estimates of BLUE's weights when the number of observations; $n=2k+1; k=2$ were computed as an illustrative example. Empirical example on BLUE's weights computation was demonstrated using four purposively selected real life data sets (each with varying sample sizes) that admit moving average process of order one. The results indicate that BLUE's weights computed using the proposed method estimate population mean with high precision than the arithmetic mean (AM) across the varying sample sizes of the four purposively selected data sets.

Key words: Moving average process, Best linear Unbiased Estimator, Arithmetic Mean

1. Introduction

One of the most important indicators of location measurements in statistics used in classical inference is the population mean (μ). Care is taken while computing the population mean because it is crucial to achieving equilibrium or stability for all other properties (Montgomery et al, 2015). The population mean is frequently estimated using the arithmetic mean, median, and mode. In contrast to the median and mode, the arithmetic mean (AM) is the most popular since it is easy to calculate, understand, and apply all of the observations in a dataset. It can also be mathematically adjusted. The AM is the best linear unbiased estimator (BLUE) for the population mean when the sequence of variables is independent and identically distributed because it is efficient, sufficient, consistent, and unbiased estimator of μ (Montgomery et al., 2015; Hogg, McKean and Craig, 2019). However, this assertion is not generally true for a sequence of correlated but identically distributed variables, $\langle X_1, X_2, \cdots, X_n \rangle$, with mean (μ), variance (σ^2), covariances $(R(k))$ and correlation (ρ_k) coefficients are given as (Chatfield, 2004; Box et al, 2016);

$$\mu = E(X_i) \forall i = 1,2,...,n \tag{1.1}$$

$$\sigma^2 = \mathrm{var}(X_i) = E[(X_i - \mu)^2] = E[X_t{}^2] - \mu^2 \tag{1.2}$$

$$R(k) = E(X_i, X_{i+k}) - \mu^2 == E(X_i, X_{i-k}) - \mu^2 \tag{1.3}$$

$$\rho_k = \frac{R(k)}{R(0)}, \ \forall\ k = 1,2,... \tag{1.4}$$

The arithmetic mean (AM) $\left(\overline{X}_n\right)$ in this scenario represented by the equal weighted linear combination of the variables shown in the mathematical structure;

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{1.5}$$

while, the vector space structural form is represented as;

$$\widetilde{X}_n = \boldsymbol{\beta}_n{}^T \mathbf{X}_n = \boldsymbol{\beta}_n \mathbf{X}_n{}^T \tag{1.6}$$

The expectation of AM is;

$$E\left[\overline{X}_n\right] = \mu \ \forall \, n \tag{1.7}$$

and variance is computed as (Chatfield, 2004);

$$\operatorname{var}(\overline{X}_n) = \ R(0)\, S\left(\boldsymbol{\beta}_n, \boldsymbol{\rho}\right) \tag{1.8}$$

$$S\left(\boldsymbol{\beta}_n, \boldsymbol{\rho}\right) = \frac{1}{n}\left[1 + \ 2\sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right)\rho_k\right] \tag{1.9}$$

for large n, $1 - \frac{k}{n} \approx 1$. Thus (Chatfield, 2004),

$$\operatorname{var}(\overline{X}_n) = \frac{R(0)}{n}\left[n \ + \ 2\sum_{k=1}^{\infty} \rho_k\right] \tag{1.10}$$

where n is the number of observations, also referred as the sample size, $\boldsymbol{\beta}_n$ is an $(n \times 1)$ vector of the sequence of coefficients; $\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}$ and $\mathbf{X}_n$ is an $(n \times 1)$ vector of correlated but identically distributed observations; $X_1, X_2, \ldots, X_n$. With a large sample size, the AM is an unbiased and reliable estimator of the mean for a series of correlated variables. When there are few observations ($n < 30$ or 40), it is not necessarily a reliable and effective estimate of the population mean (Michael, 1986; Chatfield, 2004; Box et al, 2016). The AM estimate of the population mean is questionable since, in the case of a few observations; the covariance of the data observations either increases or decreases (Allen, 1939; Michael, 1986; Hill et al, 2011; Box et al, 2016). The influence of the covariance of the data observations on the AM estimate of the mean has directed research towards determining the best unbiased estimate of the population mean. The best unbiased estimate for the mean of a short autoregressive time series and the derivation of the best linear unbiased estimate (BLUE) for the slope parameter of sequence of Chain Base Estimation (CBE) derived variables of time series are two examples of published research (Pham and Tran, 1992; Iwueze et al, 2011; Iwueze et al, 2012). However, the development of best linear unbiased estimate for the mean of correlated variables has been observed to be a herculean task because coefficients or weights of BLUE are unknown, practically difficult to compute, and analyst avoids computing the coefficients or weights but, rather use the AM even when variables are correlated (Michael, 1986; Box et al, 2016). But, having realized that the problem of using BLUE lies in the ability of practitioners to compute the coefficients or weights, this paper will develope an efficient short cut method that will enhance ease of computing the coefficients or weights of BLUE when the correlated variables have moving average process of order one covariance structure. To do this, consider generalized weighted linear combination of the correlated variables, which shall be called the linear unbiased estimator (LUE) $\left(\widetilde{X}_n\right)$ of the mean, whose mathematical structural form is represented as (John, 1965);

$$\widetilde{X}_n = \sum_{i=1}^{n} \alpha_i X_i ; \ \forall \ \alpha_i \geq 0 \tag{1.11}$$

while, the vector space structural form is represented as;

$$\widetilde{X}_n = \boldsymbol{\alpha}_n{}^T \mathbf{X}_n = \boldsymbol{\alpha}_n \mathbf{X}_n{}^T ; \boldsymbol{\alpha}_n \geq 0 \tag{1.12}$$

where $\boldsymbol{\alpha}_n$ is an $(n \times 1)$ vector of the sequence of coefficients or weights of $\widetilde{X}_n$; $\alpha_1, \alpha_2, ..., \alpha_n$, $\boldsymbol{\alpha}_n^T$ is the transpose of $\boldsymbol{\alpha}_n$, $\mathbf{X}_n$ is an $(n \times 1)$ vector of sequence of correlated observations; $X_1, X_2, ..., X_n$ and $\mathbf{X}_n^T$ is the transpose of $\mathbf{X}_n$. The expectation of $\widetilde{X}_n$ is;

$$E(\widetilde{X}_n) = \sum_{i=1}^{n} \alpha_i E(X_i) = \mu \sum_{i=1}^{n} \alpha_i \tag{1.13}$$

The estimator is expected to be unbiased. That is;

$$E(\widetilde{X}_n) = \sum_{i=1}^{n} \alpha_i E(X_i) = \mu \sum_{i=1}^{n} \alpha_i = \mu \tag{1.14}$$

Hence, the necessary condition for unbiasedness implies that

$$\sum_{i=1}^{n} \alpha_i = 1 \tag{1.15}$$

The variance of $\widetilde{X}_n$ is evaluated to be

$$\operatorname{var}(\widetilde{X}_n) = R(0) S(\boldsymbol{\alpha}_n) \tag{1.16}$$

$$S(\boldsymbol{\alpha}_n) = \sum_{i=1}^{n} \alpha_i^2 + 2 \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \rho_k \alpha_i \alpha_{i+k} \tag{1.17}$$

where $R(0)$ is the variance of the correlated sequence of observations considered to be a constant. $\widetilde{X}_n$ is a suboptimal method which is considered to be BLUE if it leads to obtaining unbiased estimate of mean with minimal possible variance as well as least computing time of assessment. For optimality of $\widetilde{X}_n$ in order to obtain minimal possible variance, first, the covariance structure of the correlated variables must be estimated. The autoregressive, moving average and autoregressive moving average covariance structures are some well-known structures for correlated observations (Chatfield, 2004; Box et al, 2016; Brockwell & Davis, 2016). As a result of these diverse covariance structures, algebraic expression for computing the estimates of $\boldsymbol{\alpha}_n$ coefficients or weights of $\widetilde{X}_n$ differs from one correlated series structure to another. Secondly, $\boldsymbol{\alpha}_n$ weights must be efficiently computed and such problem reduces to solving the quadratic programming problem posed by minimizing variance of $\widetilde{X}_n$ subject to the unbiasedness property; since the variance of the correlated sequence of observations are considered to be a constant; (Iwueze *et al*, 2015a, b). That is;

$$\left.\begin{array}{l} Min \operatorname{var}(\widetilde{X}_n) = Min\, S(\boldsymbol{\alpha}_n) \\ subject\ to\ (S.t.): \sum_{i=1}^{n} \alpha_i = 1 \\ \boldsymbol{\alpha}_n \geq 0 \end{array}\right\} \tag{1.18}$$

An attempt made to solve the resulting mathematical problem using the method of moments affirmed that it becomes tedious to obtain algebraic expressions for estimating $\boldsymbol{\alpha}_n$ weights as the number of observations increases (Iwueze *et al*, 2015b). Therefore, our target is to propose an efficient short cut method for obtaining expressions suitable for computing estimates of $\boldsymbol{\alpha}_n$ weights as the number of correlated observations increases. This proposed method which provides computational shortcuts for computing $\boldsymbol{\alpha}_n$ coefficients or weights of $\widetilde{X}_n$ with ease as the number of correlated observations increases is a major contribution of this work to the mathematical literature. Hence, this paper has been divided into sections for effective discussions on the realization of the subject matter. The materials and method is discussed in Section 2, Section 3 is devoted for results and Discussion, Section 4 discusses application while, Section 5 Concludes.

2. Materials and Method

2.1 The Covariance Structure of the Moving Average Process of Order one

The covariance structure of moving average process of order q, ($\mathrm{MA}(\mathrm{q}), q=1$) will be utilized while deriving the weights of the best linear unbiased estimator (BLUE). The structural form representation of $\mathrm{MA}(1)$ process is (Box *et al*, 2016);

$$X_t = \theta_0 - \theta_1 a_{t-1} + a_t = \theta_0 + (1-\theta_1 B)\, a_t \tag{2.1}$$

where a_t is a white noise process with zero mean and constant variance $\sigma_a{}^2$, θ_1 is a real coefficient, θ_0 is a constant and B is the backward shift operator. $\mathrm{MA}(1)$ process is always stationary but it is invertible if $|\theta_1|<1$. The expectation of $\mathrm{MA}(1)$ process is;

$$E(X_t) = \theta_0 \tag{2.2}$$

The autocovariance $(R(k))$ of (2.1) is;

$$R(k)=\begin{cases}(1+\theta_1{}^2)\sigma_a^2, & k=0\\ -\theta_1\sigma_a^2, & k=\pm 1\\ 0, & k\neq \pm 1\end{cases} \tag{2.3}$$

while, the autocorrelation (ρ_k) function of (2.1) is;

$$\rho_k=\begin{cases}1, & k=0\\ \dfrac{-\theta_1}{1+\theta_1{}^2}, & k=\pm 1\\ 0, & k\neq \pm 1\end{cases} \tag{2.4}$$

Throughout the discussion, we shall assume that $\rho_1=\rho$ while deriving the BLUE 's weights.

2.2 Pattern Recognition of Unknown Weights of Best Linear Unbiased Estimator

First, the Langrangian Multiplier Method (Humayun,1990; Inyama, 2007) was utilized in solving the quadratic programming problem (1.23) in order to compute the unknown weights $\boldsymbol{\alpha}_n=(\alpha_1,\alpha_2,\ldots,\alpha_n)^T$ of best linear unbiased estimator (BLUE) for even and odd number of observations; $n=2k$ and $n=2k+1$ at fixed $k=1,2$. Since $R(0)$ is considered to be a constant, Equation (1.23) reduces to;

$$\left.\begin{aligned}&Min\ \mathrm{var}(\widetilde{X}_n) = Min\, S(\boldsymbol{\alpha}_n)\\ &subject\ to\ (S.t.): \sum_{i=1}^{n}\alpha_i = 1\\ &\boldsymbol{\alpha}_n \geq 0\end{aligned}\right\} \tag{2.5}$$

The algorithm for the application of the Langrangian Multiplier Method is;

Step One: Re-write the quadratic programming problem (2.5) in standard condensed form as;

$$\left.\begin{aligned}&Min\ S(\boldsymbol{\alpha}_n) = \boldsymbol{\alpha}_n{}^T \boldsymbol{\rho}_n \boldsymbol{\alpha}_n\\ &subject\,to\ (S.t.):\ \mathbf{1}_n{}^T \boldsymbol{\alpha}_n = 1\\ &\boldsymbol{\alpha}_n \geq 0\end{aligned}\right\} \tag{2.6}$$

where $\boldsymbol{\alpha}_n$ is the $(n\times 1)$ vector of unknown weights for BLUE coefficients $\alpha_1,\alpha_2,\ldots,\alpha_n$, $\boldsymbol{\rho}_n$ is the $(n\times n)$ correlation matrix of the correlated random variables with moving average process of order one covariance structure and $\mathbf{1}_n$ is the $(n\times 1)$ identity vector with one as the only entries.

Step Two: Form the Langrangian Multiplier function as;

$$L(\boldsymbol{\alpha}_n,\lambda)=\boldsymbol{\alpha}_n{}^T\boldsymbol{\rho}_n\boldsymbol{\alpha}_n-\lambda(\mathbf{1}_n{}^T\boldsymbol{\alpha}_n-1)=\boldsymbol{\alpha}_n{}^T\boldsymbol{\rho}_n\boldsymbol{\alpha}_n-\lambda(\boldsymbol{\alpha}_n{}^T\mathbf{1}_n-1) \tag{2.7}$$

where λ is the Langrangian Multiplier constant and $\mathbf{1}_n{}^T\boldsymbol{\alpha}_n$ is a scalar.

Step Three: Differentiate (2.7) partially with respect to $\boldsymbol{\alpha}$ and λ using the method of Calculus, equate to zero as;

$$\frac{\partial L(\boldsymbol{\alpha}_n,\lambda)}{\partial\boldsymbol{\alpha}}=2\boldsymbol{\rho}_n\boldsymbol{\alpha}_n-\lambda\mathbf{1}_n=0 \tag{2.8}$$

$$\frac{\partial L(\boldsymbol{\alpha}_n,\lambda)}{\partial\lambda}=-(\mathbf{1}_n{}^T\boldsymbol{\alpha}_n-1)=-(\boldsymbol{\alpha}_n{}^T\mathbf{1}_n-1)=0 \tag{2.9}$$

Step Four: Solve Equations (2.8) and (2.9) simultaneously to obtain the expression for $\boldsymbol{\alpha}_n$ as $\boldsymbol{\alpha}_0$ and λ as λ_0 at $n=2k$ and $n=2k+1$ at fixed $k=1,2$;

$$\boldsymbol{\alpha}_0=\frac{\boldsymbol{\rho}_n{}^{-1}\mathbf{1}_n}{\mathbf{1}_n{}^T\boldsymbol{\rho}_n{}^{-1}\mathbf{1}_n} \tag{2.10}$$

$$\lambda_0=\frac{2}{\mathbf{1}_n{}^T\boldsymbol{\rho}_n{}^{-1}\mathbf{1}_n} \tag{2.11}$$

where $\boldsymbol{\rho}_n{}^{-1}$ is the inverse of Matrix $\boldsymbol{\rho}_n$. Computations of BLUE's weights are demonstrated for cases A through D, when the covariance structure of the correlated random variables has moving average process of order one (MA(1)) structure, in Appendices I through IV. The result in Appendices I through IV shows that the existing pattern in BLUE Weight's when $n=2k$ and $n=2k+1$ at fixed $k=1,2$ .are;

$$n=2k;\begin{cases}\alpha_1=\alpha_2;k=1\\ \begin{cases}\alpha_1=\alpha_4\\ \alpha_2=\alpha_3\end{cases};k=2\end{cases} \tag{2.12}$$

$$n=2k+1;\begin{cases}\begin{cases}\alpha_1=\alpha_3\\ \alpha_2\text{ is distinct}\end{cases};k=1\\ \begin{cases}\alpha_1=\alpha_5\\ \alpha_2=\alpha_4\\ \alpha_3\text{ is distinct}\end{cases};k=2\end{cases} \tag{2.13}$$

2.3 Pattern Generalization of the Best Linear Unbiased Estimator (BLUE) vector of coefficients

The principle of mathematical induction (see Proposition 2.1) will be utilized to establish a general pattern for the $(n\times 1)$ Best Linear Unbiased Estimator (BLUE) vector of coefficients or weights, $\boldsymbol{\alpha}_n=(\alpha_1,\alpha_2,\ldots,\alpha_n)^T$. The generalized pattern will be utilized to define symmetric relationship existing among BLUE's vector of coefficients or weights.

Proposition 2.1 (Principle of Mathematical Induction): If with each positive integer n, there is associated a statement U_n, then every statement U_n is true provided the following conditions hold:

i. U_1 is true.

ii. Whenever k is a positive integer such that U_k is true, then U_{k+1} is also true.

Proof: See Earl (1988).

Then, the generalization of the Pattern for the $(n \times 1)$ vector of coefficients or weights, $\boldsymbol{\alpha}_n = (\alpha_1, \alpha_2, ..., \alpha_n)^T$, of BLUE are derived for Case one: when n is even; $n = 2k \ \forall \ k = 1, 2$ and Case Two: when n is odd; $n = 2k + 1 \forall \ k = 1, 2$ using postulate of Proposition 2.1.

Case one: when sample size is even; $n = 2k \ \forall \ k = 1, 2$

It has been established that for such case in,

$$n = 2k; \begin{cases} \alpha_1 = \alpha_2 \, ; k = 1 \\ \begin{cases} \alpha_1 = \alpha_4 \\ \alpha_2 = \alpha_3 \end{cases} ; k = 2 \end{cases} \tag{2.14}$$

At $U_1 = 2$; $\widetilde{X}_2 = (\alpha_1, \alpha_2)$ such that $\alpha_1 = \alpha_2$ (True)

At $U_2 = 4$; $\widetilde{X}_4 = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ such that $\begin{cases} \alpha_1 = \alpha_4 \\ \alpha_2 = \alpha_3 \end{cases}$ (True)

At $U_k = 2k$; $\widetilde{X}_{2k} = (\alpha_1, \alpha_2, ..., \alpha_{2k})$ such that $\begin{cases} \alpha_1 = \alpha_{2k} = \alpha_{2k-1+1} \\ \alpha_2 = \alpha_{2k-1} = \alpha_{2k-2+1} \\ . \quad . \quad . \\ \alpha_k = \alpha_{U_k - 1} = \alpha_{2k-k+1} = \alpha_{U_k - k + 1} \end{cases}$

$\Rightarrow$ at $U_k = 2k$; $\alpha_d = \alpha_{U_k - d + 1}$; $d = 1, 2, ..., k$.

Then at $U_{k+1} = 2(k+1)$; $\widetilde{X}_{2(k+1)} = (\alpha_1, \alpha_2, ..., \alpha_{2(k+1)})$ such that

$$\begin{cases} \alpha_1 = \alpha_{2(k+1)} = \alpha_{2(k+1)-1+1} \\ \alpha_2 = \alpha_{2(k+1)-1} = \alpha_{2(k+1)-2+1} \\ . \quad . \quad . \\ \alpha_k = \alpha_{U_{k+1} - (k-1)} = \alpha_{2(k+1)-k+1} = \alpha_{U_{k+1} - k + 1} \\ \alpha_{k+1} = \alpha_{U_{k+1} - ((k+1)-1)} = \alpha_{2(k+1)-(k+1)+1} = \alpha_{U_{k+1} - (k+1)+1} \end{cases}$$

$\Rightarrow$ at $U_{k+1} = 2(k+1)$; $\alpha_d = \alpha_{U_{k+1} - d + 1}$; $d = 1, 2, ..., k+1$ *and for any* $k > 0$.

Therefore, generally for $n = 2k \ \forall$ fixed $k = 1, 2, ...$

$$\alpha_d = \alpha_{U_k - d + 1}; \ d = 1, 2, ..., k \textit{ and for any } k > 0 \tag{2.15}$$

Case Two: when sample size is odd; $n = 2k + 1 \ \forall$ fixed $k = 1, 2, ...$

The generalization for the case when the sample size is even will be established in this Section. It has been established that for case two,

$$n = 2k + 1; \begin{cases} \begin{cases} \alpha_1 = \alpha_3 \\ \alpha_2 \text{ is distinct} \end{cases} ; k = 1 \\ \begin{cases} \alpha_1 = \alpha_5 \\ \alpha_2 = \alpha_4 \\ \alpha_3 \text{ is distinct} \end{cases} ; k = 2 \end{cases} \tag{2.16}$$

At $U_1 = 3$; $\widetilde{X}_3 = (\alpha_1, \alpha_2, \alpha_3)$ such that $\begin{cases} \alpha_1 = \alpha_3 \\ \alpha_2 \text{ is distinct} \end{cases}$ (True)

At $U_2=5$; $\widetilde{X}_4=(\alpha_1,\, \alpha_2\,,\alpha_3\,,\alpha_4\,,\alpha_5)$ such that $\begin{cases}\alpha_1=\alpha_5\\ \alpha_2=\alpha_4\\ \alpha_3 \text{ is distinct}\end{cases}$ (True)

At $U_k=2k+1$; $\widetilde{X}_{2k+1}=(\alpha_1,\, \alpha_2\,,...,\alpha_{2k+1})$ such that

$$\begin{cases}\alpha_1=\alpha_{2k+1}=\alpha_{2k+1-1+1}\\ \alpha_2=\alpha_{2k+1-1}=\alpha_{2k+1-2+1}\\ . \quad . \quad .\\ \alpha_k=\alpha_{U_k-(k-1)}=\alpha_{2k+1-k+1}=\alpha_{U_k-k+1}\\ \alpha_{k+1}=\alpha_{2k-(k+1)+1}=\alpha_{U_k-(k+1)+1}=\alpha_{U_k-k} \text{ is distinct}\end{cases}$$

$\Rightarrow$ at $U_k=2k$; $\begin{cases}\alpha_d=\alpha_{U_k-d+1};\ d=1,2,...,k \ and\ for\, any\, k>0\\ \alpha_{k+1}=\alpha_{n-k} \text{ is distinct}\end{cases}$.

Then at $U_{k+1}=2(k+1)+1$; $\widetilde{X}_{2(k+1)+1}=(\alpha_1,\, \alpha_2\,,...,\alpha_{2(k+1)+1})$ such that

$$\begin{cases}\alpha_1=\alpha_{2k+3}=\alpha_{2(k+1)+1-0}\\ \alpha_2=\alpha_{2k+2}=\alpha_{2(k+1)+1-1}=\alpha_{2(k+1)+1-2+1}\\ . \quad . \quad .\\ \alpha_k=\alpha_{U_k-(k-1)}=\alpha_{2(k+1)+1-k+1}=\alpha_{U_k-k+1}\\ \alpha_{k+1}=\alpha_{U_{k+1}-((k+1)-1)}=\alpha_{2(k+1)+1-(k+1)+1}=\alpha_{2(k+1)+1-k}=\alpha_{U_{k+1}-k} \text{ is distinct}\end{cases}$$

$\Rightarrow$ at $U_{k+1}=2(k+1)+1$; $\begin{cases}\alpha_d=\alpha_{U_{k+1}-d+1};\, d=1,2,...,k+1 and\ for\, any\, k>0\\ \alpha_{k+1}=\alpha_{n-k} \text{ is distinct}\end{cases}$.

Therefore, generally for $n=2k+1 \quad \forall$ fixed $k=1,2,...$

$$\begin{cases}\alpha_d=\alpha_{n-d+1};\ d=1,2,...,k\, and\ for\, any\, k>0\\ \alpha_{k+1}=\alpha_{n-k} \text{ is distinct}\end{cases}. \tag{2.17}$$

We have established symmetric relationship in Vector Algebra using the Principle of Mathematical Induction. What is a symmetric vector? It will be defined based on whether the size of the vector is odd or even.

Definition 2.1: Let $\boldsymbol{\alpha}_n=(\alpha_1,\alpha_2,...,\alpha_n)^T$ be an $(n\times 1)$ vector of coefficients. Then, $\boldsymbol{\alpha}_n$ is said to be an even symmetric vector iff;

i. the number of observations n equal to $2k$ in size for any fixed positive interger k, $k>0$ **ii.** for any fixed positive interger k, $k>0$, the pattern of the entries or elements or coefficients of the vector are such that $\alpha_d=\alpha_{n-d+1}$; $d=1,2,...,k$. **iii.** line of symmetry passes through the mid-point of the value $\dfrac{n+1}{2}$ between the two successive coefficients α_k and α_{k+1}.

Definition 2.2: Let $\boldsymbol{\alpha}_n=(\alpha_1,\alpha_2,...,\alpha_n)^T$ be a $(n\times 1)$vector of coefficients. Then, $\boldsymbol{\alpha}_n$ is said to be an odd symmetric vector iff;

i. the number of observations n equal to $2k+1$ in size for any fixed positive interger k, $k>0$. **ii.** for any fixed positive interger k, $k>0$, the pattern of the entries or elements or coefficients of the vector are such that $\alpha_d=\alpha_{n-d+1}$; $d=1,2,...,k$ and $\alpha_{k+1}=\alpha_{n-k}$ is distinct. **iii.** line of symmetry passes through the point corresponding to the coefficient α_{k+1} for any fixed positive interger k, $k>0$.

See Figures 2.1 and 2.2 for Symmetric representation of BLUE's vector of weights when $n=2k;k=1,2,...$ and $n=2k+1;k=1,2,...$ respectively.

2.4 An Efficient Short Cut Method for Computing Weights or Coefficients of the Best Linear Unbiased Estimator (BLUE)

The proposed efficient short cut method, devised to reduce the time of assessment and eliminate computational complexity involved in computing the covariance matrix $\left(\boldsymbol{\rho}_n; \textit{for fixed}\, n\right)$ of correlated random variables as number of observations n increases, was obtained by adding the generalized pattern of BLUE's vector of coefficients or weights to quadratic programming problem (1.23) as;

$$\left.\begin{aligned}
&Min\,\text{var}\,(\tilde{X}_n) \equiv Min\ S(\boldsymbol{\alpha}_n) = \sum_{i=1}^{n}\alpha_i^{\,2} + 2\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\rho_k\,\alpha_i\,\alpha_{i+k}\\
&S.t.: \sum_{i=1}^{n}\alpha_i = 1\\
&\qquad \alpha_d = \alpha_{n-d+1}\ ; d=1,2,...,k; \textit{for any fixed}\,\text{int}\,\textit{erger}\,k \geq 1\,\textit{and}\,n\\
&\qquad \alpha_1,\alpha_2,...,\alpha_n \geq 0
\end{aligned}\right\} \tag{2.18}$$

The quadratic programming problem (2.18) reduces to standard condensed form, after substituting the pattern of the coefficients in the objective function and unbiasedness linear constraint, as;

$$\left.\begin{aligned}
&Min\ S(\boldsymbol{\alpha}_h) = \boldsymbol{\alpha}_h^{\,T}\,\rho_h\,\boldsymbol{\alpha}_h\\
&S.t.:\ \mathbf{J}_h^{\,T}\,\boldsymbol{\alpha}_h = 1\\
&\qquad \boldsymbol{\alpha}_h \geq 0\,\forall\, h
\end{aligned}\right\} \tag{2.19}$$

where $\boldsymbol{\alpha}_h$ is the $(h\times 1)$ vector of unknown weights for BLUE coefficients, $\mathbf{J}_h$ is the $(h\times 1)$ vector of constants, $\boldsymbol{\rho}_h$ is the $(h\times h)$ correlation matrix of the correlated random variables and;

$$h=\begin{cases}\dfrac{n}{2}; n=2k\ \text{for any fixed positive interger } k>0\\[2mm] \dfrac{n+1}{2}; n=2k+1\ \text{for any fixed positive interger } k>0\end{cases} \tag{2.20}$$

The Langrangian multiplier Model was obtained as;

$$L(\boldsymbol{\alpha}_h,\lambda) = \boldsymbol{\alpha}_h^{\,T}\,\boldsymbol{\rho}_h\,\boldsymbol{\alpha}_h - \lambda\left(\mathbf{J}_h^{\,T}\,\boldsymbol{\alpha}_h - 1\right) = \boldsymbol{\alpha}_h^{\,T}\,\boldsymbol{\rho}_h\,\boldsymbol{\alpha}_h - \lambda\left(\boldsymbol{\alpha}_h^{\,T}\,\mathbf{J}_h - 1\right) \tag{2.21}$$

where λ is the Langrangian Multiplier constant and $\mathbf{1}_h^{\,T}\,\boldsymbol{\alpha}_h$ is a scalar. The Langrangian Multiplier Model was minimized to obtain;

$$\frac{\partial L(\boldsymbol{\alpha}_h,\lambda)}{\partial \boldsymbol{\alpha}_h} = 2\boldsymbol{\rho}_h\,\boldsymbol{\alpha}_h - \lambda\,\mathbf{J}_h \tag{2.22}$$

$$\frac{\partial L(\boldsymbol{\alpha}_h,\lambda)}{\partial \lambda} = -\left(\mathbf{J}_h^{\,T}\,\boldsymbol{\alpha}_h - 1\right) = -\left(\boldsymbol{\alpha}_h^{\,T}\,\mathbf{J}_h - 1\right) \tag{2.23}$$

The resulting mathematical expression for obtaining the estimates of $\boldsymbol{\alpha}_h$ as $\boldsymbol{\alpha}_{h0}$ and λ were obtained as;

$$\boldsymbol{\alpha}_{h0} = \frac{\boldsymbol{\rho}_h^{\,-1}\,\mathbf{J}_h}{\mathbf{J}_h^{\,T}\boldsymbol{\rho}_h^{\,-1}\,\mathbf{J}_h} \tag{2.24}$$

$$\lambda_0 = \frac{2}{\mathbf{J}_h^{\,T}\boldsymbol{\rho}_h^{\,-1}\,\mathbf{J}_h} \tag{2.25}$$

where $\boldsymbol{\rho}_h^{-1}$ is the inverse of Matrix $\boldsymbol{\rho}_h$ and λ_0 is the Langrangian constant. For a moving average process of order one, the vector of constants and correlation matrix of the correlated random variables when the sample size is even; $n=2k$ and $h=k$ for any fixed positive interger $k>0$, is given as;

$$\mathbf{J}_h = [2 \quad 2 \quad . \quad . \quad . \quad 2]^T \tag{2.26}$$

and

$$\boldsymbol{\rho}_h = \begin{bmatrix} 2 & 2\rho & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 2\rho & 2 & 2\rho & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 2\rho & 2 & 2\rho & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\rho & 2 & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 2 & 2\rho & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 2\rho & 2 & 2\rho & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 2\rho & 2 & 2\rho \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 2\rho & 2(1+\rho) \end{bmatrix} \tag{2.27}$$

while, the vector of constants and correlation matrix of the correlated random variables, when $n=2k+1$ and $h=k+1$ for any positive interger $k>0$, are given as;

$$\mathbf{J}_h = [2 \quad 2 \quad . \quad . \quad . \quad 2 \quad 1]^T \tag{2.28}$$

and

$$\boldsymbol{\rho}_h = \begin{bmatrix} 2 & 2\rho & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 2\rho & 2 & 2\rho & 0 & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 2\rho & 2 & 2\rho & . & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\rho & 2 & . & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 2 & 2\rho & 0 & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 2\rho & 2 & 2\rho & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 2\rho & 2 & 2\rho \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 2\rho & 1 \end{bmatrix} \tag{2.29}$$

where ρ is the correlation of the correlated variables with moving average structure of order one. Then, demonstration on the applicability of the proposed method to compute the algebraic expressions of the weights or coefficients of BLUE are shown in Appendices V and VI for $n=2k$ and $n=2k+1$at fixed $k=2$. An illustrative example on computation of the estimates of the weights or coefficients of BLUE were also executed for $n=2k+1$at fixed $k=2$.

3.0 Results and Discussion

When the sample size (n) is even or odd; $n=2k \ or \ n=2k+1 \forall k=1,2,...,8$., Tables 4.1 and 4.2 display the algebraic formulae for calculating the coefficients or weights of the best linear unbiased estimator (BLUE) for correlated variables that admits the covariance structure of the moving average process of order one. The findings in Tables 4.1 and 4.2 demonstrate that the covariance of the correlated variables with the moving average process structure determines the coefficients or weights of BLUE. When the sample size is even, the vector of coefficients or weights of the best linear unbiased estimator (BLUE) is even symmetric in nature. When the sample size is odd, the vector of coefficients or weights of BLUE is odd symmetric in nature. In addition, the weights of the two estimating techniques (arithmetic mean (AM) and Best Linear Unbiased Estimator (BLUE)) are equal term by term when the correlation value is zero and at sample size equal to two. The estimation of BLUE weights or coefficients with sample size $n = 2k+1$at fixed $k = 2$ are shown in Table 4.3. For a sequence of correlated variables with an MA(1) covariance structure, the results in Table 4.3 demonstrate that the sum of the estimates of BLUE's weights or coefficients is unity, indicating that each set of values at a particular correlation point leads to the realization of an unbiased estimate of the population mean. The variance of BLUE and AM at corresponding correlation values and constant sample size of five; $n = 2k+1$at fixed $k = 2$; are displayed in Table 4.4. With the exception of the instances where the correlational value's significance is in doubt, Table 4.4's data demonstrate that BLUE has higher precision than AM.

4.0 APPLICATION

Empirical demonstration on the use of the proposed efficient short cut method for computing weights of best linear unbiased estimator (BLUE) of the population mean of correlated variables was illustrated using real life data that admits moving average process of order one (MA(1)). The real life data is the monthly average exchange of Naira per unit of EURO currency for the period January 2004 to December 2018 (CBN, 2021). The first order non-seasonal differenced series to remove the source of non-stationarity due to mean, also regarded as the stationary series due to non-presence of other sources of non-stationarity, admits MA(1) process whose parameter estimates, residual sum of squares and Modified Box-Pierce (Ljung-Box) Chi-Square Statistic evaluated using Minitab 22 series Software are shown as:

Final Estimates of Parameters

Type	Coef	SE Coef	T-Value	P-Value
MA 1	-0.4001	0.0687	-5.82	0.000

Differencing: 1 regular difference

Residual Sums of Squares

DF	SS	MS
178	9761.46	54.8397

Back forecasts excluded

Modified Box-Pierce (Ljung-Box) Chi-Square Statistic

Lag	12	24	36	48
Chi-Square	5.84	16.76	21.04	29.37
DF	11	23	35	47
P-Value	0.884	0.821	0.970	0.979

The modified Box-Pierce (Ljung-Box) Chi-Square test statistic and p-value at the seasonal lags; 12, 24, 36 and 48, are not significant because the modified Box-Pierce (Ljung-Box) Chi-Square test statistic is less than $\chi^2(\ln n)$ tabulated value and p-value at the seasonal lags is greater than 0.05 level of significant. Therefore, the hypothesis that the residual is independent and identically distributed is not rejected. Similarly, the p value of the estimated coefficient of the MA(1) model is less than 0.05 level of significant indicating that the estimated coefficient contributes significantly to the behavioral of the MA(1) model. Hence, the computation of the BLUE"s weights for correlated variables that admits the covariance structure of MA(1) model was demonstrated by purposively selecting four different data sets shown in Table 4.5, with varying number of observations equal to ten, eleven, twelve and thirteen, from the real life series. Estimates of autocorrelation coefficients across the varying sample sizes for the four purposively selected real life data sets shown in Table 4.6 indicate that there is a cut off at lag one of the autocorrelation coefficients. The result in Table 4.6 indicates that all four selected real life data sets admit moving average process of order one. Hence, the computation of the estimates of BLUE's weights and population mean with its associated variance was evaluated following the procedural steps;

i. Select the initial starting value for the estimate of the autocorrelation coefficient $(\hat{\rho})$ of the sequence of correlated variables, $\langle X_t, t=1,2,\ldots,n\rangle$. The best estimate for $\hat{\rho}$ is selected by considering that $E(\tilde{X}_n)=E(\bar{X}_n)$ under the null hypothesis and computing the value of the autocovariance (d_k) and/or autocorrelation $(\hat{\rho})$ coefficients of the correlated variables with;

$$d_k=\frac{1}{n-k}\sum_{t=1}^{n}\left(X_t-E(X_t)\right)\left(X_{t+k}-E(X_{t+k})\right), k=0,1,\ldots \quad (4.1)$$

$$\hat{\rho}_k=\frac{d_k}{d_0}, k=0,1,\ldots \quad (4.2)$$

$$\Rightarrow \hat{\rho}_1=\hat{\rho}=\frac{d_1}{d_0} \quad (4.3)$$

ii. Evaluate BLUE’s algebraic expressions at the varying number of observations (n) with $\hat{\rho}$ for selected correlated variables, $\langle X_t, t=1,2,\ldots,n\rangle$ in order to obtain the estimates of BLUE’s weights at each n.

iii. Compute the population mean and variance of the correlated variables, $\langle X_t, t=1,2,\ldots,n\rangle$ with the computed estimates of BLUE’s weights in step ii.

The results in Table 4.7 indicate that the use of AM weights to compute the mean over-estimate the mean of the correlated variables with low precision, while BLUE weights computed using the new proposed efficient method estimate the mean with high precision across the varying sample sizes of the four purposively selected data sets.

5.0 Conclusion

An efficient shortcut method for calculating the algebraic weights or coefficients of BLUE has been proposed in the literature. The proposed effective technique suggested was used to calculate the algebraic weights or coefficients of the best linear unbiased estimator (BLUE) for correlated variables with the covariance structure of a moving average process of order one when the number of observations (n) is even or odd; $n=2k$ *and* $n=2k+1 \forall k=1,2,\ldots,8$. The proposed method decreases both the evaluation time and the computing complexity involved in determining the covariance matrix of BLUE's vector of weights as the number of observations rises. An illustrative example to show the applicability of the efficient method in computing the weights or coefficients of BLUE was demonstrated for $n=2k+1 \forall k=2$. The result from the example shows that BLUE’s weights lead to obtaining reliable and efficient estimates of mean at significance values of the correlation. Empirical example on the computation of BLUE’s weights was demonstrated using four purposively selected data sets, each with varying sample sizes, from a real life data that admits moving average process of order one. The use of computed BLUEs weights estimate the mean with high precision, while the use of AM weights overestimate the mean of the correlated variables with low precision across all varying sample sizes of the four purposively selected data sets. Therefore, the efficient short cut method is recommended to practitioners for use in computing the algebraic weights or coefficients of BLUE.

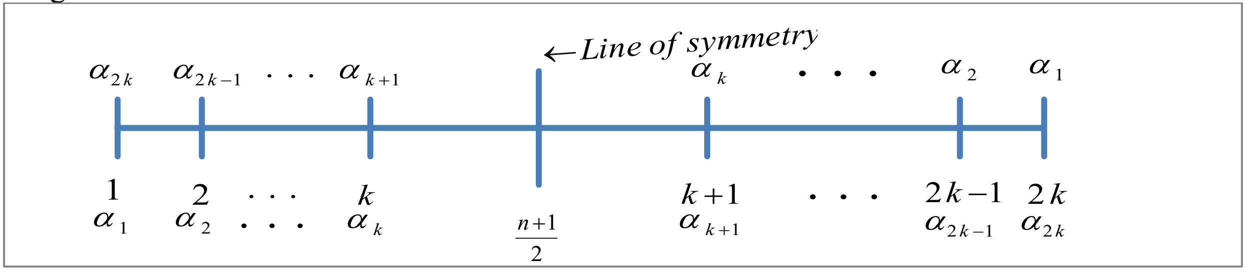

Figure 2.1: Symmetric Representation of BLUE's vector of weights when $n=2k; k=1,2,...$ Note that $\alpha_d = \alpha_{n-d+1}$; $d=1,2,...,k, for\, any\, fixed\, k > 0$ and the line of symmetry passes through the mid-point value between two successive coefficients α_k and α_{k+1}

Table 4.1: Algebraic Expression of BLUE's coefficients or weights when sample size $n=2k, h=n/2$ and $h=k \ \forall k=1,2,...,8$

k	BLUE coefficients or weights
1	$\boldsymbol{\alpha}_1 = \begin{bmatrix}\alpha_1 = \alpha_2\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\end{bmatrix}$
2	$\boldsymbol{\alpha}_2 = \begin{bmatrix}\alpha_1 = \alpha_4 \\ \alpha_2 = \alpha_3\end{bmatrix} = \frac{1}{2(\rho-2)}\begin{bmatrix}-1 \\ \rho-1\end{bmatrix}$
3	$\boldsymbol{\alpha}_3 = \begin{bmatrix}\alpha_1 = \alpha_6 \\ \alpha_2 = \alpha_5 \\ \alpha_3 = \alpha_4\end{bmatrix} = \frac{1}{4\rho^2+4\rho-6}\begin{bmatrix}\rho^2-1 \\ \rho^2+\rho-1 \\ \rho-1\end{bmatrix}$
4	$\boldsymbol{\alpha}_4 = \begin{pmatrix}\alpha_1 = \alpha_8 \\ \alpha_2 = \alpha_7 \\ \alpha_3 = \alpha_6 \\ \alpha_4 = \alpha_5\end{pmatrix} = \frac{1}{4\rho^3-12\rho^2-6\rho+8}\begin{pmatrix}1-2\rho^2 \\ 1-\rho-2\rho^2+\rho^3 \\ 1-\rho-\rho^2+\rho^3 \\ 1-\rho-\rho^2\end{pmatrix}$
5	$\boldsymbol{\alpha}_5 = \begin{pmatrix}\alpha_1 = \alpha_{10} \\ \alpha_2 = \alpha_9 \\ \alpha_3 = \alpha_8 \\ \alpha_4 = \alpha_7 \\ \alpha_5 = \alpha_6\end{pmatrix} = \frac{1}{\left[10-8\rho-24\rho^2+12\rho^3+6\rho^4\right]}\begin{pmatrix}1-3\rho^2+\rho^4 \\ 1-\rho-3\rho^2+2\rho^3+\rho^4 \\ 1-\rho-2\rho^2+2\rho^3 \\ 1-\rho-2\rho^2+\rho^3 \\ 1-\rho-2\rho^2+\rho^3+\rho^4\end{pmatrix}$
6	$\boldsymbol{\alpha}_6 = \begin{bmatrix}\alpha_1 = \alpha_{12} \\ \alpha_2 = \alpha_{11} \\ \alpha_3 = \alpha_{10} \\ \alpha_4 = \alpha_9 \\ \alpha_5 = \alpha_8 \\ \alpha_6 = \alpha_7\end{bmatrix} = \frac{1}{\begin{bmatrix}12-10\rho-40\rho^2+24\rho^3 \\ +24\rho^4-6\rho^5\end{bmatrix}}\begin{bmatrix}1-4\rho^2+3\rho^4 \\ 1-\rho-4\rho^2+3\rho^3+3\rho^4-\rho^5 \\ 1-\rho-3\rho^2+3\rho^3+\rho^4-\rho^5 \\ 1-\rho-3\rho^2+2\rho^3+\rho^4 \\ 1-\rho-3\rho^2+2\rho^3+2\rho^4 \\ 1-\rho-3\rho^2+2\rho^3+2\rho^4-\rho^5\end{bmatrix}$
7	$\boldsymbol{\alpha}_7 = \begin{bmatrix}\alpha_1 = \alpha_{14} \\ \alpha_2 = \alpha_{13} \\ \alpha_3 = \alpha_{12} \\ \alpha_4 = \alpha_{11} \\ \alpha_5 = \alpha_{10} \\ \alpha_6 = \alpha_9 \\ \alpha_7 = \alpha_8\end{bmatrix} = \frac{1}{\begin{bmatrix}14-12\rho-60\rho^2+40\rho^3 \\ +60\rho^4-24\rho^5-8\rho^6\end{bmatrix}}\begin{bmatrix}1-5\rho^2+6\rho^4-\rho^6 \\ 1-\rho-5\rho^2+4\rho^3+6\rho^4-3\rho^5-\rho^6 \\ 1-\rho-4\rho^2+4\rho^3+3\rho^4-3\rho^5 \\ 1-\rho-4\rho^2+3\rho^3+3\rho^4-\rho^5 \\ 1-\rho-4\rho^2+3\rho^3+4\rho^4-\rho^5-\rho^6 \\ 1-\rho-4\rho^2+3\rho^3+4\rho^4-2\rho^5-\rho^6 \\ 1-\rho-4\rho^2+3\rho^3+4\rho^4-2\rho^5\end{bmatrix}$

8	$\boldsymbol{\alpha}_8=\begin{bmatrix}\alpha_1=\alpha_{14}\\ \alpha_2=\alpha_{13}\\ \alpha_3=\alpha_{12}\\ \alpha_4=\alpha_{11}\\ \alpha_5=\alpha_{10}\\ \alpha_6=\alpha_9\\ \alpha_7=\alpha_8\\ \alpha_7=\alpha_8\end{bmatrix}=\dfrac{1}{\begin{bmatrix}16-14\rho-84\rho^2+60\rho^3\\ +120\rho^4-60\rho^5-40\rho^6\\ +6\rho^7\end{bmatrix}}\begin{bmatrix}1-6\rho^2+10\rho^4-4\rho^6\\ 1-\rho-6\rho^2+5\rho^3+10\rho^4-6\rho^5-4\rho^6+\rho^7\\ 1-\rho-5\rho^2+5\rho^3+6\rho^4-6\rho^5-\rho^6+\rho^7\\ 1-\rho-5\rho^2+4\rho^3+6\rho^4-3\rho^5-\rho^6\\ 1-\rho-5\rho^2+4\rho^3+7\rho^4-3\rho^5-3\rho^6-\rho^7\\ 1-\rho-5\rho^2+4\rho^3+7\rho^4-4\rho^5-3\rho^6+\rho^7\\ 1-\rho-5\rho^2+4\rho^3+7\rho^4-4\rho^5-2\rho^6+\rho^7\\ 1-\rho-5\rho^2+4\rho^3+7\rho^4-4\rho^5-2\rho^6\end{bmatrix}$

Table 4.2: Algebraic Expression of BLUE's coefficients or weights when sample size $n=2k+1, h=(n+1)/2 \quad \forall k=1,2,\ldots,8.$

k	BLUE coefficients or weights
1	$\boldsymbol{\alpha}_2=\begin{bmatrix}\alpha_1=\alpha_3\\ \alpha_2\end{bmatrix}=\dfrac{1}{-(3-4\rho)}\begin{bmatrix}\rho-1\\ 2\rho-1\end{bmatrix}$
2	$\boldsymbol{\alpha}_3=\begin{bmatrix}\alpha_1=\alpha_5\\ \alpha_2=\alpha_4\\ \alpha_3\end{bmatrix}=\dfrac{1}{-(5-8\rho-\rho^2)}\begin{bmatrix}\rho^2+\rho-1\\ 2\rho-1\\ -(\rho-1)^2\end{bmatrix}$
3	$\boldsymbol{\alpha}_4=\begin{pmatrix}\alpha_1=\alpha_7\\ \alpha_2=\alpha_6\\ \alpha_3=\alpha_5\\ \alpha_4\end{pmatrix}=\dfrac{1}{7-12\rho-6\rho^2+8\rho^3}\begin{pmatrix}1-\rho-2\rho^2+\rho^3\\ 1-2\rho-\rho^2+2\rho^3\\ 1-2\rho+\rho^3\\ 1-2\rho\end{pmatrix}$
4	$\boldsymbol{\alpha}_5=\begin{pmatrix}\alpha_1=\alpha_9\\ \alpha_2=\alpha_8\\ \alpha_3=\alpha_7\\ \alpha_4=\alpha_6\\ \alpha_5\end{pmatrix}=\dfrac{1}{9-16\rho-15\rho^2+24\rho^3+\rho^4}\begin{pmatrix}1-\rho-3\rho^2+2\rho^3+\rho^4\\ 1-2\rho-2\rho^2+4\rho^3\\ 1-2\rho-\rho^2+3\rho^3-\rho^4\\ 1-2\rho-\rho^2+2\rho^3\\ 1-2\rho-\rho^2+2\rho^3+\rho^4\end{pmatrix}$
5	$\boldsymbol{\alpha}_6=\begin{pmatrix}\alpha_1=\alpha_{11}\\ \alpha_2=\alpha_{10}\\ \alpha_3=\alpha_9\\ \alpha_4=\alpha_8\\ \alpha_5=\alpha_7\\ \alpha_6\end{pmatrix}=\dfrac{1}{\begin{bmatrix}11-20\rho-28\rho^2+48\rho^3\\ +9\rho^4-12\rho^5\end{bmatrix}}\begin{pmatrix}1-\rho-4\rho^2+3\rho^3+3\rho^4-\rho^5\\ 1-2\rho-3\rho^2+6\rho^3+\rho^4-2\rho^5\\ 1-2\rho-2\rho^2+5\rho^3-\rho^4-\rho^5\\ 1-2\rho-2\rho^2+4\rho^3\\ 1-2\rho-2\rho^2+4\rho^3+\rho^4-\rho^5\\ 1-2\rho-2\rho^2+4\rho^3+\rho^4-2\rho^5\end{pmatrix}$

6	$\boldsymbol{\alpha}_7=\begin{pmatrix}\alpha_1=\alpha_{13}\\ \alpha_2=\alpha_{12}\\ \alpha_3=\alpha_{11}\\ \alpha_4=\alpha_{10}\\ \alpha_5=\alpha_9\\ \alpha_6=\alpha_8\\ \alpha_7\end{pmatrix}=\dfrac{1}{\begin{bmatrix}13-24\rho-45\rho^2+80\rho^3\\ +30\rho^4-48\rho^5-\rho^6\end{bmatrix}}\begin{pmatrix}1-\rho-5\rho^2+4\rho^3+6\rho^4-3\rho^5-\rho^6\\ 1-2\rho-4\rho^2+8\rho^3+3\rho^4-6\rho^5\\ 1-2\rho-3\rho^2+7\rho^3-4\rho^5+\rho^6\\ 1-2\rho-3\rho^2+6\rho^3+\rho^4-2\rho^5\\ 1-2\rho-3\rho^2+6\rho^3+2\rho^4-3\rho^5-\rho^6\\ 1-2\rho-3\rho^2+6\rho^3+2\rho^4-4\rho^5\\ 1-2\rho-3\rho^2+6\rho^3+2\rho^4-4\rho^5+\rho^6\end{pmatrix}$
7	$\boldsymbol{\alpha}_8=\begin{pmatrix}\alpha_1=\alpha_{15}\\ \alpha_2=\alpha_{14}\\ \alpha_3=\alpha_{13}\\ \alpha_4=\alpha_{12}\\ \alpha_5=\alpha_{11}\\ \alpha_6=\alpha_{10}\\ \alpha_7=\alpha_9\\ \alpha_8\end{pmatrix}=\dfrac{1}{\begin{bmatrix}15-28\rho-66\rho^2\\ +120\rho^3+66\rho^4\\ -120\rho^5-4\rho^6\\ +16\rho^7\end{bmatrix}}\begin{pmatrix}1-\rho-6\rho^2+5\rho^3+10\rho^4-6\rho^5-4\rho^6+\rho^7\\ 1-2\rho-5\rho^2+10\rho^3+5\rho^4-12\rho^5+\rho^6+2\rho^7\\ 1-2\rho-4\rho^2+9\rho^3+2\rho^4-9\rho^5+2\rho^6+\rho^7\\ 1-2\rho-4\rho^2+8\rho^3+2\rho^4-6\rho^5+2\rho^6\\ 1-2\rho-4\rho^2+8\rho^3+4\rho^4-7\rho^5-2\rho^6+\rho^7\\ 1-2\rho-4\rho^2+8\rho^3+4\rho^4-8\rho^5-\rho^6+2\rho^7\\ 1-2\rho-4\rho^2+8\rho^3+4\rho^4-8\rho^5+\rho^7\\ 1-2\rho-4\rho^2+8\rho^3+4\rho^4-8\rho^5\end{pmatrix}$

Table 4.2 continues: Algebraic Expression of BLUE's coefficients or weights when sample size $n=2k+1, h=(n+1)/2 \quad \forall k=1,2,\ldots,8.$

K	BLUE coefficients or weights
8	$\boldsymbol{\alpha}_9=\begin{pmatrix}\alpha_1=\alpha_{17}\\ \alpha_2=\alpha_{16}\\ \alpha_3=\alpha_{15}\\ \alpha_4=\alpha_{14}\\ \alpha_5=\alpha_{13}\\ \alpha_6=\alpha_{12}\\ \alpha_7=\alpha_{11}\\ \alpha_8=\alpha_{10}\\ \alpha_9\end{pmatrix}=\dfrac{1}{\begin{bmatrix}17-32\rho-91\rho^2\\ +168\rho^3+135\rho^4\\ -240\rho^5-50\rho^6\\ +76\rho^7+\rho^8\end{bmatrix}}\begin{pmatrix}1-\rho-7\rho^2+6\rho^3+15\rho^4-10\rho^5-10\rho^6+4\rho^7+\rho^8\\ 1-2\rho-6\rho^2+12\rho^3+10\rho^4-20\rho^5-4\rho^6+7\rho^7\\ 1-2\rho-5\rho^2+11\rho^3+5\rho^4-16\rho^5+2\rho^6+5\rho^7-\rho^8\\ 1-2\rho-5\rho^2+10\rho^3+6\rho^4-12\rho^5-\rho^6+2\rho^7\\ 1-2\rho-5\rho^2+10\rho^3+7\rho^4-13\rho^5-4\rho^6+4\rho^7+\rho^8\\ 1-2\rho-5\rho^2+10\rho^3+7\rho^4-14\rho^5-3\rho^6+6\rho^7\\ 1-2\rho-5\rho^2+10\rho^3+7\rho^4-14\rho^5-2\rho^6+4\rho^7-\rho^8\\ 1-2\rho-5\rho^2+10\rho^3+7\rho^4-14\rho^5-2\rho^6+4\rho^7\\ 1-2\rho-5\rho^2+10\rho^3+7\rho^4-14\rho^5-2\rho^6+4\rho^7+\rho^8\end{pmatrix}$

Table 4.3: Estimates of the weights or coefficients of BLUE for $n=2k+1, k=2$

ρ	α_1	α_2	α_3	α_4	α_5	$\sum_{i=1}^{5}\alpha_i$	ρ	α_1	α_2	α_3	α_4	α_5	$\sum_{i=1}^{5}\alpha_i$
-0.50	0.1429	0.2286	0.2571	0.2286	0.1429	1	-0.20	0.1768	0.2134	0.2195	0.2134	0.1768	1
-0.49	0.1440	0.2281	0.2558	0.2281	0.1440	1	-0.19	0.1780	0.2128	0.2184	0.2128	0.1780	1
-0.48	0.1451	0.2277	0.2544	0.2277	0.1451	1	-0.18	0.1791	0.2122	0.2173	0.2122	0.1791	1
-0.47	0.1463	0.2272	0.2531	0.2272	0.1463	1	-0.17	0.1802	0.2117	0.2162	0.2117	0.1802	1
-0.46	0.1474	0.2267	0.2517	0.2267	0.1474	1	-0.16	0.1814	0.2111	0.2151	0.2111	0.1814	1
-0.45	0.1486	0.2263	0.2504	0.2263	0.1486	1	-0.15	0.1825	0.2104	0.2141	0.2104	0.1825	1
-0.44	0.1497	0.2258	0.2490	0.2258	0.1497	1	-0.14	0.1837	0.2098	0.2130	0.2098	0.1837	1
-0.43	0.1508	0.2253	0.2477	0.2253	0.1508	1	-0.13	0.1848	0.2092	0.2120	0.2092	0.1848	1
-0.42	0.1520	0.2248	0.2464	0.2248	0.1520	1	-0.12	0.1860	0.2086	0.2110	0.2086	0.1860	1
-0.41	0.1531	0.2244	0.2451	0.2244	0.1531	1	-0.11	0.1871	0.2079	0.2100	0.2079	0.1871	1
-0.40	0.1542	0.2239	0.2438	0.2239	0.1542	1	-0.10	0.1883	0.2073	0.2090	0.2073	0.1883	1
-0.39	0.1554	0.2234	0.2425	0.2234	0.1554	1	-0.09	0.1894	0.2066	0.2080	0.2066	0.1894	1
-0.38	0.1565	0.2229	0.2412	0.2229	0.1565	1	-0.08	0.1906	0.2059	0.2070	0.2059	0.1906	1
-0.37	0.1576	0.2224	0.2399	0.2224	0.1576	1	-0.07	0.1917	0.2052	0.2061	0.2052	0.1917	1
-0.36	0.1588	0.2219	0.2386	0.2219	0.1588	1	-0.06	0.1929	0.2045	0.2052	0.2045	0.1929	1
-0.35	0.1599	0.2214	0.2374	0.2214	0.1599	1	-0.05	0.1941	0.2038	0.2043	0.2038	0.1941	1
-0.34	0.1610	0.2209	0.2361	0.2209	0.1610	1	-0.04	0.1952	0.2031	0.2034	0.2031	0.1952	1
-0.33	0.1621	0.2204	0.2349	0.2204	0.1621	1	-0.03	0.1964	0.2023	0.2025	0.2023	0.1964	1
-0.32	0.1633	0.2199	0.2336	0.2199	0.1633	1	-0.02	0.1976	0.2016	0.2016	0.2016	0.1976	1
-0.31	0.1644	0.2194	0.2324	0.2194	0.1644	1	-0.01	0.1988	0.2008	0.2008	0.2008	0.1988	1
-0.30	0.1655	0.2189	0.2312	0.2189	0.1655	1	0.00	0.2000	0.2000	0.2000	0.2000	0.2000	1
-0.29	0.1667	0.2184	0.2300	0.2184	0.1667	1	0.01	0.2012	0.1992	0.1992	0.1992	0.2012	1
-0.28	0.1678	0.2178	0.2288	0.2178	0.1678	1	0.02	0.2024	0.1984	0.1984	0.1984	0.2024	1
-0.27	0.1689	0.2173	0.2276	0.2173	0.1689	1	0.03	0.2036	0.1975	0.1977	0.1975	0.2036	1
-0.26	0.1700	0.2168	0.2264	0.2168	0.1700	1	0.04	0.2049	0.1966	0.1970	0.1966	0.2049	1
-0.25	0.1712	0.2162	0.2252	0.2162	0.1712	1	0.05	0.2061	0.1958	0.1963	0.1958	0.2061	1
-0.24	0.1723	0.2157	0.2241	0.2157	0.1723	1	0.06	0.2073	0.1948	0.1956	0.1948	0.2073	1
-0.23	0.1734	0.2151	0.2229	0.2151	0.1734	1	0.07	0.2086	0.1939	0.1950	0.1939	0.2086	1
-0.22	0.1746	0.2146	0.2218	0.2146	0.1746	1	0.08	0.2098	0.1929	0.1944	0.1929	0.2098	1
-0.21	0.1757	0.2140	0.2206	0.2140	0.1757	1	0.09	0.2111	0.1920	0.1938	0.1920	0.2111	1

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Table 4.3 continues: Estimates of the weights or coefficients of BLUE for $n=2k+1, k=2$

ρ	α_1	α_2	α_3	α_4	α_5	$\sum_{i=1}^{n}\alpha_i$	ρ	α_1	α_2	α_3	α_4	α_5	$\sum_{i=1}^{n}\alpha_i$
0.10	0.2124	0.1909	0.1933	0.1909	0.2124	1	0.31	0.2450	0.1568	0.1964	0.1568	0.2450	1
0.11	0.2137	0.1899	0.1928	0.1899	0.2137	1	0.32	0.2471	0.1540	0.1978	0.1540	0.2471	1
0.12	0.2150	0.1888	0.1924	0.1888	0.2150	1	0.33	0.2493	0.1510	0.1994	0.1510	0.2493	1
0.13	0.2164	0.1877	0.1920	0.1877	0.2164	1	0.34	0.2515	0.1478	0.2013	0.1478	0.2515	1
0.14	0.2177	0.1865	0.1916	0.1865	0.2177	1	0.35	0.2539	0.1444	0.2034	0.1444	0.2539	1
0.15	0.2191	0.1853	0.1913	0.1853	0.2191	1	0.36	0.2564	0.1407	0.2058	0.1407	0.2564	1
0.16	0.2204	0.1841	0.1910	0.1841	0.2204	1	0.37	0.2591	0.1366	0.2086	0.1366	0.2591	1
0.17	0.2218	0.1828	0.1908	0.1828	0.2218	1	0.38	0.2620	0.1322	0.2117	0.1322	0.2620	1
0.18	0.2233	0.1814	0.1906	0.1814	0.2233	1	0.39	0.2650	0.1273	0.2153	0.1273	0.2650	1
0.19	0.2247	0.1800	0.1905	0.1800	0.2247	1	0.40	0.2683	0.1220	0.2195	0.1220	0.2683	1
0.20	0.2262	0.1786	0.1905	0.1786	0.2262	1	0.41	0.2719	0.1160	0.2243	0.1160	0.2719	1
0.21	0.2277	0.1771	0.1905	0.1771	0.2277	1	0.42	0.2758	0.1093	0.2298	0.1093	0.2758	1
0.22	0.2292	0.1755	0.1906	0.1755	0.2292	1	0.43	0.2801	0.1018	0.2363	0.1018	0.2801	1
0.23	0.2308	0.1738	0.1908	0.1738	0.2308	1	0.44	0.2848	0.0933	0.2438	0.0933	0.2848	1
0.24	0.2324	0.1720	0.1911	0.1720	0.2324	1	0.45	0.2902	0.0835	0.2526	0.0835	0.2902	1
0.25	0.2340	0.1702	0.1915	0.1702	0.2340	1	0.46	0.2963	0.0722	0.2631	0.0722	0.2963	1
0.26	0.2357	0.1683	0.1920	0.1683	0.2357	1	0.47	0.3033	0.0589	0.2756	0.0589	0.3033	1
0.27	0.2375	0.1662	0.1926	0.1662	0.2375	1	0.48	0.3115	0.0430	0.2909	0.0430	0.3115	1
0.28	0.2393	0.1641	0.1933	0.1641	0.2393	1	0.49	0.3213	0.0238	0.3097	0.0238	0.3213	1
0.29	0.2411	0.1618	0.1942	0.1618	0.2411	1	0.50	0.3333	0.0000	0.3333	0.0000	0.3333	1
0.30	0.2430	0.1594	0.1952	0.1594	0.2430	1							

Table 4.4: Variance of BLUE and Arithmetic Mean for $n=2k+1, k=2$

ρ	$\text{var}(\tilde{X}_n)$	$\text{var}(\bar{X}_n)$	ρ	$\text{var}(\tilde{X}_n)$	$\text{var}(\bar{X}_n)$	ρ	$\text{var}(\tilde{X}_n)$	$\text{var}(\bar{X}_n)$	ρ	$\text{var}(\tilde{X}_n)$	$\text{var}(\bar{X}_n)$
-0.50	0.0286	0.0400	-0.28	0.1068	0.1104	-0.06	0.1806	0.1808	0.16	0.2499	0.2512
-0.49	0.0322	0.0432	-0.27	0.1102	0.1136	-0.05	0.1839	0.1840	0.17	0.2529	0.2544
-0.48	0.0359	0.0464	-0.26	0.1137	0.1168	-0.04	0.1871	0.1872	0.18	0.2559	0.2576
-0.47	0.0395	0.0496	-0.25	0.1171	0.1200	-0.03	0.1904	0.1904	0.19	0.2589	0.2608
-0.46	0.0431	0.0528	-0.24	0.1205	0.1232	-0.02	0.1936	0.1936	0.20	0.2619	0.2640
-0.45	0.0467	0.0560	-0.23	0.1240	0.1264	-0.01	0.1968	0.1968	0.21	0.2649	0.2672
-0.44	0.0503	0.0592	-0.22	0.1274	0.1296	0.00	0.2000	0.2000	0.22	0.2678	0.2704
-0.43	0.0539	0.0624	-0.21	0.1308	0.1328	0.01	0.2032	0.2032	0.23	0.2708	0.2736
-0.42	0.0575	0.0656	-0.20	0.1341	0.1360	0.02	0.2064	0.2064	0.24	0.2737	0.2768
-0.41	0.0611	0.0688	-0.19	0.1375	0.1392	0.03	0.2096	0.2096	0.25	0.2766	0.2800
-0.40	0.0647	0.0720	-0.18	0.1409	0.1424	0.04	0.2127	0.2128	0.26	0.2795	0.2832
-0.39	0.0682	0.0752	-0.17	0.1443	0.1456	0.05	0.2159	0.2160	0.27	0.2824	0.2864
-0.38	0.0718	0.0784	-0.16	0.1476	0.1488	0.06	0.2190	0.2192	0.28	0.2852	0.2896
-0.37	0.0753	0.0816	-0.15	0.1510	0.1520	0.07	0.2222	0.2224	0.29	0.2880	0.2928
-0.36	0.0789	0.0848	-0.14	0.1543	0.1552	0.08	0.2253	0.2256	0.30	0.2908	0.2960
-0.35	0.0824	0.0880	-0.13	0.1576	0.1584	0.09	0.2284	0.2288	0.31	0.2936	0.2992
-0.34	0.0859	0.0912	-0.12	0.1609	0.1616	0.10	0.2315	0.2320	0.32	0.2964	0.3024
-0.33	0.0894	0.0944	-0.11	0.1642	0.1648	0.11	0.2346	0.2352	0.33	0.2991	0.3056
-0.32	0.0929	0.0976	-0.10	0.1675	0.1680	0.12	0.2377	0.2384	0.34	0.3018	0.3088
-0.31	0.0964	0.1008	-0.09	0.1708	0.1712	0.13	0.2407	0.2416	0.35	0.3045	0.3120
-0.30	0.0999	0.1040	-0.08	0.1741	0.1744	0.14	0.2438	0.2448	0.36	0.3071	0.3152
-0.29	0.1033	0.1072	-0.07	0.1774	0.1776	0.15	0.2469	0.2480	0.37	0.3097	0.3184

Table 4.4 continues: Variance of BLUE and Arithmetic Mean for $n=2k+1, k=2$

ρ	$\text{var}(\tilde{X}_n)$	$\text{var}(\bar{X}_n)$
0.38	0.3122	0.3216
0.39	0.3147	0.3248
0.40	0.3171	0.3280
0.41	0.3194	0.3312
0.42	0.3217	0.3344
0.43	0.3238	0.3376
0.44	0.3259	0.3408
0.45	0.3278	0.3440
0.46	0.3295	0.3472
0.47	0.3310	0.3504
0.48	0.3322	0.3536
0.49	0.3330	0.3568
0.50	0.3333	0.3600

Table 4.5: Four purposively selected data sets on monthly average exchange of Naira per unit of EURO currency (X_t)

Data 1			Data 2			Data 3			Data 4		
Year	Month	X_t	Year	Month	X_t	Year	Month	X_t	Year	Month	X_t
2004	November	172.67	2010	July	189.83	2017	May	337.72	2011	April	220.08
2004	December	177.95	2010	August	191.9	2017	June	343.24	2011	May	219.66
2005	January	174.13	2010	September	195.91	2017	July	358.50	2011	June	220.22
2005	February	173.05	2010	October	208.34	2017	August	360.93	2011	July	216.08
2005	March	176.26	2010	November	203.64	2017	September	364.53	2011	August	216.79
2005	April	171.8	2010	December	197.27	2017	October	359.34	2011	September	211.73
2005	May	168.52	2011	January	200.57	2017	November	359.07	2011	October	208.22
2005	June	161.67	2011	February	205.58	2017	December	362.36	2011	November	208.78
2005	July	160.00	2011	March	211.17	2018	January	373.00	2011	December	206.52
2005	August	162.81	2011	April	220.08	2018	February	377.84	2012	January	202.52
			2011	May	219.66	2018	March	377.19	2012	February	206.71
						2018	April	374.22	2012	March	206.05
									2012	April	205.00

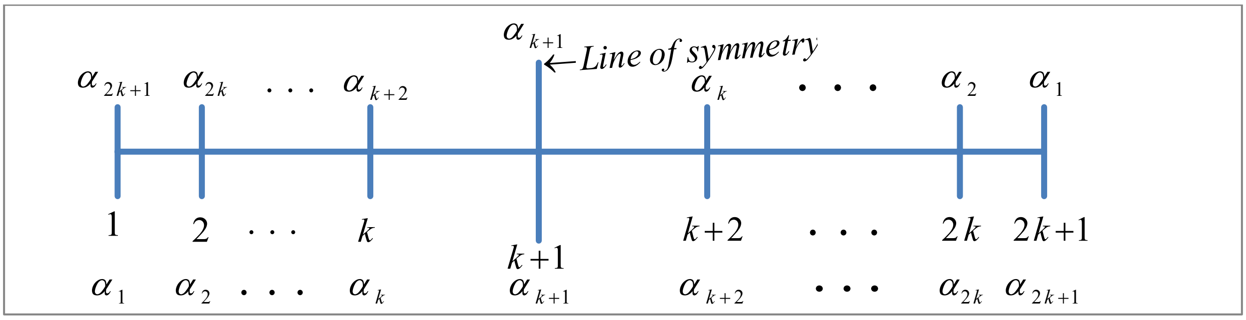

Figure 2.2: Symmetric Representation of BLUE's vector of weights when $n=2k+1; k=1,2,\ldots$

Note that $\alpha_d = \alpha_{n-d+1}; d=1,2,\dots,k$, $\alpha_{k+1} = \alpha_{n-k}$ is distinct and the line of line of symmetry passes through the point corresponding to the coefficient α_{k+1} for any fixed positive interger k , $k>0$.

Table 4.6: Estimates of Autocorrelation Coefficients across the sample sizes for Data sets 1, 2, 3 and 4

Sample Size	lags	Estimates of Autocorrelation Coefficients	T value
n	k	$\hat{\rho}_k$	
10	**1**	**0.7239**	**2.29**
	2	**0.3248**	**0.72**
	3	**0.0066**	**0.01**
11	**1**	**0.6032**	**2.00**
	2	**0.1333**	**0.34**
	3	**-0.0925**	**-0.23**
12	**1**	**0.6490**	**2.25**
	2	**0.2420**	**0.62**
	3	**0.0309**	**0.08**
13	**1**	**0.7800**	**2.81**
	2	**0.5904**	**1.43**
	3	**0.3743**	**0.79**

Table 4.7: Estimates of BLUE Weights, Arithmetic mean $\left(\overline{X}_n\right)$ with the associated variance, and Mean using BLUE $\left(\widetilde{X}_n\right)$ with its associated variance

Sample size	Estimates of Autocorrelation **Coefficients**	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	$\overline{X}_n$	$\widetilde{X}_n$	$v\left(\overline{X}_n\right)$	$v\left(\widetilde{X}_n\right)$
n	$\hat{\rho}_k$																	
10	**0.7239**	**0.1372**	**0.1211**	**0.0061**	**0.1811**	**0.0544**	**0.0544**	**0.1811**	**0.0061**	**0.1211**	**0.1372**				**169.8860**	**169.82866**	**7.88453**	**7.7000**
11	**0.6032**	**0.1713**	**0.0174**	**0.1012**	**0.1162**	**0.0076**	**0.1726**	**0.0076**	**0.1162**	**0.1012**	**0.0174**	**0.1713**			**203.9955**	**203.7284**	**13.1388**	**12.5311**
12	**0.6490**	**0.1260**	**0.0798**	**0.0250**	**0.1557**	**0.0091**	**0.1043**	**0.1043**	**0.0091**	**0.1557**	**0.0250**	**0.0798**	**0.1260**		**362.3300**	**361.4900**	**22.3517**	**21.7771**
13	**0.7800**	**0.1090**	**0.0959**	**0.0037**	**0.1350**	**0.0589**	**0.0252**	**0.1445**	**0.0252**	**0.0589**	**0.1350**	**0.0037**	**0.0959**	**0.1090**	**211.4100**	**210.8900**	**5.2553**	**5.1464**

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REFERENCES

Allen,T. C. (1939). On the Mathematics of the Representative Method of Sampling. The Annals of Mathematical Statistics, 10: 26-34.

Box, G. E. P, Jenkins, G. M., Reinsel, G. C. & Ljung, G. M. (2016). Time Series Analysis, Forecasting and Control, fifth edition, John Wiley & Sons, Inc., Hoboken, New Jersey

Brockwell, P. J. & Davis, R. A. (2016). Introduction to Time Series and Forecasting, Third Edition, Springer International Publishing, Switzerland. http://extras.springer.com. DOI: 10.1007/978-3-319-29854-2.

CBN (Central Bank of Nigeria) (2021). https://www.cbn.gov.ng/rates/exrate.asp?year=2021

Chatfield, C. (2004). The Analysis of Time Series: An Introduction, Sixth Edition. London: Chapman and Hall. DOI: 10.22237/jmasm/1209615240

Earl, W. S. (1988). Calculus with Analytic Geometry, fourth edition, PWS – Kent Publishing Company, USA.

Hill, R. C., Griffiths, W. E. and Lim, G. C. (2011). Principles of Econometrics, Fourth edition, John Wiley and sons Inc., USA. ISBN: 978 0 470 62673 3.

Hogg, R. V., McKean, J. W. & Craig, A. T. (2019). Introduction to Mathematical Statistics, eighth edition, Pearson Education Inc., United States of America

Humayun, R. (1990). Quadratic programming. Master of Philosophy in Statistics dissertation, Department Of Statistics, Aligarh Muslim University Aligarh, India.

Inyama, S. C. (2007). Operations Research: An Introduction. Supreme Publishers, Owerri, Nigeria. ISBN: 978 – 978 – 079 – 763 - 8

Iwueze, I. S, Nwogu, E. C & Iwu, H. C. (2012). Best linear unbiased estimate of linear trend-cycle based on FBE-Derived variables. Journal of Mathematical Computing Science, 2 (2), 189-205. ISSN: 1927-5307

Iwueze, I. S., Nwogu, E. C. & Ajaraogu, J. C. (2011) . Best linear unbiased estimate using buys-ballot procedure when trend-cycle component is linear, CBN Journal of Applied Statistics, 2 (1), 15-30. ISSN 2476-8472.

Iwueze, I. S., Okereke, E. O. & Ganiyu, A. S. (2015a). Computation of Weights of the BLUE for the Mean of Correlated Random Variables using R. Document retrieved on November 7, 2024 from academia .edu. https:// www.academia.edu.

Iwueze, I. S., Okereke, E. W. & Ganiyu, A. S. (2015b). On the Consequences of using the Arithmetic Mean as opposed to the Blue for the Estimation of the Mean of Correlated Random Variables. Conference paper presented at the Nigeria Statistical Association Conference at Benue State, Nigeria.

John, I. M. (1965). The Construction of Good Linear Unbiased Estimates from the Best Linear Estimates for a Smaller Sample Size. Technometrics, American Statistical Association and American Society for Quality Stable, 7 (4), 543-552. http://www.jstor.org/stable/1266394

Michael, E. M. (1986). Efficiencies of Weighted Averages in Stationary Autoregressive Processes. Journal of the American Statistical Association, 81: 730 – 735.

Montgomery, D. C., Jennings, C. L. and Kulahci, M. (2015). Introduction to Time Series Analysis and Forecasting, second edition, John Wiley & Sons, Inc., Hoboken, New Jersey, USA.

Pham, T. D. & Tran, L.T. (1992). On the best unbiased estimate for the mean of a short autoregressive time series, Econometric Theory, 8: 120-126.

APPENDIX I

Case A: When number of observations; $n = 2k$ at fixed $k = 1$.

$$Min\ S(\boldsymbol{\alpha}_n) = \sum_{t=1}^{2} \alpha_t^{\ 2} + 2\rho\alpha_1\alpha_2 \qquad \text{(i)}$$

$$S.t.:\ \alpha_1 + \alpha_2 = 1 \qquad \text{(ii)}$$

$$L(\boldsymbol{\alpha}_n, \lambda) = \sum_{t=1}^{2} \alpha_t^{\ 2} + 2\rho\alpha_1\alpha_2 - \lambda\left(\sum_{t=1}^{2} \alpha_t - 1\right) \qquad \text{(iii)}$$

$$= \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} + \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{bmatrix}0 & \rho\\ \rho & 0\end{bmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} - \lambda\left[\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} - 1\right] \qquad \text{(iv)}$$

Using the right and left distributive laws of Matrix arithmetic, (iv) reduces to;

$$L(\boldsymbol{\alpha}_n, \lambda) = \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} - \lambda\left[\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} - 1\right] \qquad \text{(v)}$$

$$= \begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} - \lambda\left[\begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} - 1\right] \qquad \text{(vi)}$$

The identity (vii) and correlation matrix (viii) of the quadratic form for BLUE's vector of coefficients were obtained from (vi) as;

$$\mathbf{1}_2 = \begin{pmatrix}1 & 1\end{pmatrix}^T \qquad \text{(vii)}$$

$$\boldsymbol{\rho}_2 = \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix} \qquad \text{(viii)}$$

$$|\boldsymbol{\rho}_2| \text{ or determinant of } \boldsymbol{\rho}_2 = 1 - \rho^2 \qquad \text{(ix)}$$

$$\text{Adjoint of } \boldsymbol{\rho}_2 = Adj(\boldsymbol{\rho}_2) = \begin{bmatrix}1 & -\rho\\ -\rho & 1\end{bmatrix} \qquad \text{(x)}$$

$$\boldsymbol{\rho}_2^{\ -1} = \frac{1}{1-\rho^2}\begin{bmatrix}1 & -\rho\\ -\rho & 1\end{bmatrix} \qquad \text{(xi)}$$

$$\mathbf{1}_2^{\ T}\boldsymbol{\rho}_2^{\ -1}\mathbf{1}_2 = \frac{1}{1-\rho^2}\left[\begin{pmatrix}1 & 1\end{pmatrix}\begin{bmatrix}1 & -\rho\\ -\rho & 1\end{bmatrix}\begin{pmatrix}1\\ 1\end{pmatrix}\right] = \frac{2}{1+\rho} \qquad \text{(xii)}$$

$$\boldsymbol{\rho}_2^{\ -1}\mathbf{1}_2 = \frac{1}{1-\rho^2}\begin{bmatrix}1 & -\rho\\ -\rho & 1\end{bmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{1-\rho^2}\begin{bmatrix}1-\rho\\ 1-\rho\end{bmatrix} \qquad \text{(xiii)}$$

$$\boldsymbol{\alpha}_{20} = \frac{\boldsymbol{\rho}_2^{\ -1}\mathbf{1}_2}{\mathbf{1}_2^{\ T}\boldsymbol{\rho}_2^{\ -1}\mathbf{1}_2} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\end{pmatrix}^T \qquad \text{(xiv)}$$

$$\Rightarrow \qquad \alpha_1 = \alpha_2 = \frac{1}{2} \qquad \text{(xv)}$$

APPENDIX II

Case B: When number of observations; $n = 2k$ at fixed $k = 2$

$$Min\ S(\boldsymbol{\alpha}_n) = \sum_{t=1}^{4} \alpha_t^{\ 2} + 2\rho(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4) \qquad \text{(i)}$$

$$S.t.:\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1 \qquad \text{(ii)}$$

$$L(\boldsymbol{\alpha}_n, \lambda) = \sum_{t=1}^{4} \alpha_t^{\ 2} + 2\rho(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4) - \lambda\left(\sum_{t=1}^{4} \alpha_t - 1\right) \qquad \text{(iii)}$$

APPENDIX II CONTINUES

$$= \left[\begin{array}{l} \left(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\right)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{pmatrix} \\ +\left(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\right)\begin{bmatrix}0&\rho&0&0\\\rho&0&\rho&0\\0&\rho&0&\rho\\0&0&\rho&0\end{bmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{pmatrix}-\lambda\left[\left(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\right)\begin{pmatrix}1\\1\\1\\1\end{pmatrix}-1\right] \end{array} \right]\Bigg\} \tag{iv}$$

Using the right and left distributive laws of Matrix arithmetic, (iv) reduces to;

$$L(\boldsymbol{\alpha}_n,\lambda)=\left(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\right)\begin{bmatrix}1&\rho&0&0\\\rho&1&\rho&0\\0&\rho&1&\rho\\0&0&\rho&1\end{bmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{pmatrix}-\lambda\left[\left(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\right)\begin{pmatrix}1\\1\\1\\1\end{pmatrix}-1\right] \tag{v}$$

$$=\left(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\right)\begin{bmatrix}1&\rho&0&0\\\rho&1&\rho&0\\0&\rho&1&\rho\\0&0&\rho&1\end{bmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{pmatrix}-\lambda\left[\left(1 \quad 1 \quad 1 \quad 1\right)\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\\\alpha_4\end{pmatrix}-1\right] \tag{vi}$$

The identity (vii) and correlation matrix (viii) of the quadratic form for BLUE's vector of coefficients were obtained from (vi) as;

$$\mathbf{1}_4 = \left(1 \quad 1 \quad 1 \quad 1\right)^T \tag{vii}$$

$$\boldsymbol{\rho}_4 = \begin{bmatrix}1&\rho&0&0\\\rho&1&\rho&0\\0&\rho&1&\rho\\0&0&\rho&1\end{bmatrix} \tag{viii}$$

$$\left|\boldsymbol{\rho}_4\right| \text{or determinant of } \boldsymbol{\rho}_4 = 1-3\rho^2+\rho^4 \tag{ix}$$

$$\text{Adjoint of } \boldsymbol{\rho}_4 = Adj\left(\boldsymbol{\rho}_4\right)=\begin{bmatrix}1-2\rho^2 & -\rho\left(1-\rho^2\right) & \rho^2 & -\rho^3\\ -\rho\left(1-\rho^2\right) & 1-\rho^2 & -\rho & \rho^2\\ \rho^2 & -\rho & 1-\rho^2 & -\rho\left(1-\rho^2\right)\\ -\rho^3 & \rho^2 & -\rho\left(1-\rho^2\right) & 1-2\rho^2\end{bmatrix} \tag{x}$$

$$\boldsymbol{\rho}_4^{-1}=\frac{1}{1-3\rho^2+\rho^4}\begin{bmatrix}1-2\rho^2 & -\rho\left(1-\rho^2\right) & \rho^2 & -\rho^3\\ -\rho\left(1-\rho^2\right) & 1-\rho^2 & -\rho & \rho^2\\ \rho^2 & -\rho & 1-\rho^2 & -\rho\left(1-\rho^2\right)\\ -\rho^3 & \rho^2 & -\rho\left(1-\rho^2\right) & 1-2\rho^2\end{bmatrix} \tag{xi}$$

$$\mathbf{1}_4^T\boldsymbol{\rho}_4^{-1}\mathbf{1}_4=\frac{4-6\rho-2\rho^2+2\rho^3}{1-3\rho^2+\rho^4} \tag{x}$$

$$\boldsymbol{\rho}_4^{-1}\mathbf{1}_4=\frac{1}{1-3\rho^2+\rho^4}\left[1-\rho-\rho^2 \quad 1-2\rho-\rho^3 \quad 1-2\rho-\rho^3 \quad 1-\rho-\rho^2\right]^T \tag{xi}$$

$$\boldsymbol{\alpha}_{40}=\frac{\boldsymbol{\rho}_4^{-1}\mathbf{1}_4}{\mathbf{1}_4^T\boldsymbol{\rho}_4^{-1}\mathbf{1}_4}=\frac{1}{2(\rho-2)}\left[-1 \quad \rho-1 \quad \rho-1 \quad -1\right]^T \tag{xii}$$

$$\Rightarrow \quad \left.\begin{array}{l} \alpha_1 = \alpha_4 \\ \alpha_2 = \alpha_3 \end{array}\right\} \tag{xiii}$$

APPENDIX III

Case C: When number of observations; $n = 2k+1$ at fixed $k = 1$.

$$Min\ S(\boldsymbol{\alpha}_n) = \sum_{t=1}^{3} \alpha_t^{\ 2} + 2\rho(\alpha_1\alpha_2 + \alpha_2\alpha_3) \tag{i}$$

$$S.t.:\ \alpha_1 + \alpha_2 + \alpha_3 = 1 \tag{ii}$$

$$L(\boldsymbol{\alpha}_n, \lambda) = \sum_{t=1}^{3} \alpha_t^{\ 2} + 2\rho(\alpha_1\alpha_2 + \alpha_2\alpha_3) - \lambda\left(\sum_{t=1}^{3}\alpha_t - 1\right) \tag{iii}$$

$$= \left.\left[\begin{array}{l} (\alpha_1 \quad \alpha_2 \quad \alpha_3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \\ + (\alpha_1 \quad \alpha_2 \quad \alpha_3)\begin{bmatrix} 0 & \rho & 0 \\ \rho & 0 & \rho \\ 0 & \rho & 0 \end{bmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} - \lambda\left[(\alpha_1 \quad \alpha_2 \quad \alpha_3)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 1\right] \end{array}\right]\right\} \tag{iv}$$

Using the right and left distributive laws of Matrix arithmetic, Equation (iv) reduces to;

$$L(\boldsymbol{\alpha}_n, \lambda) = (\alpha_1 \quad \alpha_2 \quad \alpha_3)\begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} - \lambda\left[(\alpha_1 \quad \alpha_2 \quad \alpha_3)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - 1\right] \tag{v}$$

$$= (\alpha_1 \quad \alpha_2 \quad \alpha_3)\begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} - \lambda\left[(1 \quad 1 \quad 1)\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} - 1\right] \tag{vi}$$

The identity (vii) and correlation matrix (viii) of the quadratic form for BLUE's vector of coefficients were obtained from (vi) as;

$$\mathbf{1}_3 = (1 \quad 1 \quad 1)^T \tag{vii}$$

$$\boldsymbol{\rho}_3 = \begin{bmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix} \tag{viii}$$

$$|\boldsymbol{\rho}_3| \text{ or determinant of } \boldsymbol{\rho}_3 = 1 - 2\rho^2 \tag{ix}$$

$$\text{Adjoint of } \boldsymbol{\rho}_3 = Adj(\boldsymbol{\rho}_3) = \begin{bmatrix} 1-\rho^2 & -\rho & \rho^2 \\ -\rho & 1 & -\rho \\ \rho^2 & -\rho & 1-\rho^2 \end{bmatrix} \tag{x}$$

$$\boldsymbol{\rho}_3^{\ -1} = \frac{1}{1-2\rho^2}\begin{bmatrix} 1-\rho^2 & -\rho & \rho^2 \\ -\rho & 1 & -\rho \\ \rho^2 & -\rho & 1-\rho^2 \end{bmatrix} \tag{xi}$$

$$\mathbf{1}_3^{\ T}\boldsymbol{\rho}_3^{\ -1}\mathbf{1}_3 = \frac{3-4\rho}{1-2\rho^2} \tag{xii}$$

$$\boldsymbol{\rho}_3^{\ -1}\mathbf{1}_3 = \frac{1}{1-2\rho^2}\left[1-\rho \quad 1-2\rho \quad 1-\rho\right]^T \tag{xiii}$$

APPENDIX III CONTINUES

$$\boldsymbol{\alpha}_{40} = \frac{\boldsymbol{\rho}_4^{-1}\mathbf{1}_4}{\mathbf{1}_4^T\boldsymbol{\rho}_4^{-1}\mathbf{1}_4} = \frac{1}{3-4\rho}\,\left[1-\rho \quad 1-2\rho \quad 1-\rho\right]^T \tag{xiv}$$

$$\Rightarrow \quad \alpha_1 = \alpha_3 \ \textit{and} \ \alpha_2 \text{ is distinct} \tag{xv}$$

APPENDIX IV

Case D: When number of observations; $n = 2k+1$ at fixed $k=2$.

$$Min\ S(\boldsymbol{\alpha}_n) = \sum_{t=1}^{5}\alpha_t^{\,2} + 2\rho(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 + + \alpha_4\alpha_5) \tag{i}$$

$$S.t.:\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1 \tag{ii}$$

$$L(\boldsymbol{\alpha}_n, \lambda) = \sum_{t=1}^{5}\alpha_t^{\,2} + 2\rho(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_4 + + \alpha_4\alpha_5) - \lambda\left(\sum_{t=1}^{5}\alpha_t - 1\right) \tag{iii}$$

$$L(\boldsymbol{\alpha}_n, \lambda) = \left[\begin{array}{l} (\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5)\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} \\ +(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5)\begin{bmatrix} 0 & \rho & 0 & 0 & 0 \\ \rho & 0 & \rho & 0 & 0 \\ 0 & \rho & 0 & \rho & 0 \\ 0 & 0 & \rho & 0 & \rho \\ 0 & 0 & 0 & \rho & 0 \end{bmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} \\ -\lambda\left[(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 1\right] \end{array}\right] \tag{iv}$$

Using the right and left distributive laws of Matrix arithmetic, Equation (iv) reduces to;

$$L(\boldsymbol{\alpha}_n, \lambda) = \left[\begin{array}{l} (\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5)\begin{bmatrix} 1 & \rho & 0 & 0 & 0 \\ \rho & 1 & \rho & 0 & 0 \\ 0 & \rho & 1 & \rho & 0 \\ 0 & 0 & \rho & 1 & \rho \\ 0 & 0 & 0 & \rho & 1 \end{bmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} \\ -\lambda\left[(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - 1\right] \end{array}\right] \tag{v}$$

APPENDIX IV CONTINUES

$$=\left[\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5\end{pmatrix}\begin{bmatrix}1 & \rho & 0 & 0 & 0\\ \rho & 1 & \rho & 0 & 0\\ 0 & \rho & 1 & \rho & 0\\ 0 & 0 & \rho & 1 & \rho\\ 0 & 0 & 0 & \rho & 1\end{bmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix}-\lambda\left[\begin{pmatrix}1 & 1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\end{pmatrix}-1\right]\right]\Bigg\} \qquad (vi)$$

The identity (vii) and correlation matrix (viii) of the quadratic form for BLUE's vector of coefficients were obtained from (vi) as;

$$\mathbf{1}_5=\begin{pmatrix}1 & 1 & 1 & 1 & 1\end{pmatrix}^T \qquad \text{(vii)}$$

$$\boldsymbol{\rho}_5=\begin{bmatrix}1 & \rho & 0 & 0 & 0\\ \rho & 1 & \rho & 0 & 0\\ 0 & \rho & 1 & \rho & 0\\ 0 & 0 & \rho & 1 & \rho\\ 0 & 0 & 0 & \rho & 1\end{bmatrix} \qquad \text{(viii)}$$

$$\left|\boldsymbol{\rho}_5\right| \text{ or determinant of } \boldsymbol{\rho}_5 = 1-4\rho^2+3\rho^4 \qquad \text{(ix)}$$

$$Adj\left(\boldsymbol{\rho}_5\right)=\begin{bmatrix}1-3\rho^2+\rho^4 & -\rho\left(1-2\rho^2\right) & \rho^2\left(1-\rho^2\right) & -\rho^3 & \rho^4\\ -\rho\left(1-2\rho^2\right) & 1-2\rho^2 & -\rho\left(1-\rho^2\right) & \rho^2 & -\rho^3\\ \rho^2\left(1-\rho^2\right) & -\rho\left(1-\rho^2\right) & \left(1-\rho^2\right)^2 & -\rho\left(1-\rho^2\right) & \rho^2\left(1-\rho^2\right)\\ -\rho^3 & \rho^2 & -\rho\left(1-\rho^2\right) & 1-2\rho^2 & -\rho\left(1-2\rho^2\right)\\ \rho^4 & -\rho^3 & \rho^2\left(1-\rho^2\right) & -\rho\left(1-2\rho^2\right) & 1-3\rho^2+\rho^4\end{bmatrix} \qquad \text{(x)}$$

$$\boldsymbol{\rho}_5^{-1}=\frac{1}{\begin{bmatrix}1-4\rho^2\\ +3\rho^4\end{bmatrix}}\begin{bmatrix}1-3\rho^2+\rho^4 & -\rho\left(1-2\rho^2\right) & \rho^2\left(1-\rho^2\right) & -\rho^3 & \rho^4\\ -\rho\left(1-2\rho^2\right) & 1-2\rho^2 & -\rho\left(1-\rho^2\right) & \rho^2 & -\rho^3\\ \rho^2\left(1-\rho^2\right) & -\rho\left(1-\rho^2\right) & \left(1-\rho^2\right)^2 & -\rho\left(1-\rho^2\right) & \rho^2\left(1-\rho^2\right)\\ -\rho^3 & \rho^2 & -\rho\left(1-\rho^2\right) & 1-2\rho^2 & -\rho\left(1-2\rho^2\right)\\ \rho^4 & -\rho^3 & \rho^2\left(1-\rho^2\right) & -\rho\left(1-2\rho^2\right) & 1-3\rho^2+\rho^4\end{bmatrix} \qquad (xi)$$

$$\mathbf{1}_5^T\boldsymbol{\rho}_5^{-1}\mathbf{1}_5=\frac{5-8\rho-6\rho^2+8\rho^3+\rho^4}{1-4\rho^2+3\rho^4} \qquad \text{(xii)}$$

$$\boldsymbol{\rho}_5^{-1}\mathbf{1}_5=\frac{1}{1-4\rho^2+3\rho^4}\begin{pmatrix}1-\rho-2\rho^2+\rho^3+\rho^4\\ 1-2\rho-\rho^2+2\rho^3\\ 1-2\rho+2\rho^3-\rho^4\\ 1-2\rho-\rho^2+2\rho^3\\ 1-\rho-2\rho^2+\rho^3+\rho^4\end{pmatrix} \qquad \text{(xiii)}$$

$$\boldsymbol{\alpha}_{40}=\frac{\boldsymbol{\rho}_4^{-1}\mathbf{1}_4}{\mathbf{1}_4^T\boldsymbol{\rho}_4^{-1}\mathbf{1}_4}=\frac{1}{5-8\rho-6\rho^2+8\rho^3+\rho^4}\begin{pmatrix}1-\rho-2\rho^2+\rho^3+\rho^4\\ 1-2\rho-\rho^2+2\rho^3\\ 1-2\rho+2\rho^3-\rho^4\\ 1-2\rho-\rho^2+2\rho^3\\ 1-\rho-2\rho^2+\rho^3+\rho^4\end{pmatrix}$$

APPENDIX IV CONTINUES

$$= \frac{1}{5-8\rho+\rho^2}\begin{pmatrix} 1-\rho+\rho^2 \\ 2\rho-1 \\ -(\rho-1)^2 \\ 2\rho-1 \\ 1-\rho+\rho^2 \end{pmatrix} \qquad \text{(xiv)}$$

$$\Rightarrow \left.\begin{array}{l} \alpha_1=\alpha_5 \\ \alpha_2=\alpha_4 \\ \alpha_3 \text{ is distinct} \end{array}\right\} \qquad \text{(xv)}$$

APPENDIX V

When number of observations; $n=2k$ at fixed $k=2$

$$Min\ S(\boldsymbol{\alpha}_n)=\sum_{t=1}^{4}\alpha_t^2+2\rho(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_4) \qquad \text{(i)}$$

$$S.t.:\ \alpha_1+\alpha_2+\alpha_3+\alpha_4=1 \qquad \text{(ii)}$$

$$\alpha_1=\alpha_4 \qquad \text{(iii)}$$

$$\alpha_2=\alpha_4 \qquad \text{(iv)}$$

Substituting (iii) and (iv) into (i) and (ii), the Langrangian Function is obtained as;

$$L(\boldsymbol{\alpha}_h,\lambda)=2\sum_{t=1}^{2}\alpha_t^2+2\rho\alpha_2^2+4\rho\alpha_1\alpha_2-\lambda\left(2\sum_{t=1}^{2}\alpha_t-1\right) \qquad \text{(v)}$$

$$=\left[\begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}+\begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{bmatrix}0 & 2\rho\\ 2\rho & 2\rho\end{bmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}-\lambda\left[\begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{pmatrix}2\\ 2\end{pmatrix}-1\right]\right]\Bigg\} \qquad \text{(iv)}$$

Using the right and left distributive laws of Matrix arithmetic, (iv) reduces to;

$$L(\boldsymbol{\alpha}_h,\lambda)=\left[\begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{bmatrix}2 & 2\rho\\ 2\rho & 2(1+\rho)\end{bmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}-\lambda\left[\begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix}\begin{pmatrix}2\\ 2\end{pmatrix}-1\right]\right] \qquad \text{(v)}$$

The identity (vi) and correlation matrix (vii) of the quadratic form for BLUE's vector of coefficients were obtained from (vi) as;

$$\mathbf{J}_2=\begin{pmatrix}2\\ 2\end{pmatrix} \qquad \text{(vi)}$$

$$\boldsymbol{\rho}_2=\begin{bmatrix}2 & 2\rho\\ 2\rho & 2(1+\rho)\end{bmatrix} \qquad \text{(vii)}$$

$$|\boldsymbol{\rho}_2| \text{ or determinant of } \boldsymbol{\rho}_2=4(1+\rho-\rho^2) \qquad \text{(viii)}$$

$$\text{Adjoint of } \boldsymbol{\rho}_2=Adj(\boldsymbol{\rho}_2)=\begin{bmatrix}2(1+\rho) & -2\rho\\ -2\rho & 2\end{bmatrix} \qquad \text{(ix)}$$

$$\boldsymbol{\rho}_2^{-1}=\frac{1}{2(1+\rho-\rho^2)}\begin{bmatrix}1+\rho & -\rho\\ -\rho & 1\end{bmatrix} \qquad \text{(x)}$$

$$\mathbf{J}_2^T\boldsymbol{\rho}_2^{-1}\mathbf{J}_2=\frac{2\ (2-\rho)}{1+\rho-\rho^2} \qquad \text{(xi)}$$

$$\boldsymbol{\rho}_2^{-1}\mathbf{J}_2=\frac{1}{1+\rho-\rho^2}\begin{bmatrix}1\\ 1-\rho\end{bmatrix} \qquad \text{(xii)}$$

APPENDIX V CONTINUES

$$\boldsymbol{\alpha}_{20}=\frac{\boldsymbol{\rho}_2^{\;-1}\mathbf{J}_2}{\mathbf{J}_2^{\;T}\boldsymbol{\rho}_2^{\;-1}\mathbf{J}_2}=\frac{1}{2\,(2-\rho)}\begin{bmatrix}1\\1-\rho\end{bmatrix} \tag{xiii}$$

$$\Rightarrow\quad \left.\begin{aligned}\alpha_1=\alpha_4&=\frac{1}{2\,(2-\rho)}\\ \alpha_2=\alpha_3&=\frac{1-\rho}{2\,(2-\rho)}\end{aligned}\right\} \tag{xiv}$$

APPENDIX VI

When number of observations; $n=2k+1$ at fixed $k=2$

$$Min\ S(\boldsymbol{\alpha}_n)=\sum_{t=1}^{5}\alpha_t^{\;2}+2\rho(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_4+\alpha_4\alpha_5) \tag{i}$$

$$S.t.:\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=1 \tag{ii}$$

$$\alpha_1=\alpha_5 \tag{iii}$$

$$\alpha_2=\alpha_4 \tag{iv}$$

Substituting (iii) and (iv) into (i) and (ii), the Langrangian Function is obtained as;

$$L(\boldsymbol{\alpha}_h,\lambda)=2\sum_{t=1}^{2}\alpha_t^{\;2}+\alpha_3^{\;2}+4\rho[\alpha_1\alpha_2+\alpha_2\alpha_3]-\lambda\left(2\sum_{t=1}^{2}\alpha_t+\alpha_3-1\right) \tag{v}$$

$$=\left.\left[\begin{array}{l}(\alpha_1\quad\alpha_2\quad\alpha_3)\begin{bmatrix}2&0&0\\0&2&0\\0&0&1\end{bmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}+(\alpha_1\quad\alpha_2\quad\alpha_3)\begin{bmatrix}0&2\rho&0\\2\rho&0&2\rho\\0&2\rho&0\end{bmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}\\ -\lambda\left[(\alpha_1\quad\alpha_2\quad\alpha_3)\begin{pmatrix}2\\2\\1\end{pmatrix}-1\right]\end{array}\right]\right\} \tag{iv}$$

Using the right and left distributive laws of Matrix arithmetic, (iv) reduces to;

$$L(\boldsymbol{\alpha}_h,\lambda)=(\alpha_1\quad\alpha_2\quad\alpha_3)\begin{bmatrix}2&2\rho&0\\2\rho&2&2\rho\\0&2\rho&1\end{bmatrix}\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix}-\lambda\left[(\alpha_1\quad\alpha_2\quad\alpha_3)\begin{pmatrix}2\\2\\1\end{pmatrix}-1\right] \tag{v}$$

The identity (vi) and correlation matrix (vii) of the correlated random variables were obtained from (vi) as;

$$\mathbf{J}_3=\begin{pmatrix}2\\2\\1\end{pmatrix} \tag{vi}$$

$$\boldsymbol{\rho}_3=\begin{bmatrix}2&2\rho&0\\2\rho&2&2\rho\\0&2\rho&1\end{bmatrix} \tag{vii}$$

$$|\boldsymbol{\rho}_3|\,\text{or determinant of}\ \boldsymbol{\rho}_3=4(1-3\rho^2) \tag{viii}$$

$$\text{Adjoint of}\ \boldsymbol{\rho}_3=Adj(\boldsymbol{\rho}_3)=\begin{bmatrix}2(1-2\rho^2)&-2\rho&4\rho^2\\-2\rho&2&-4\rho\\4\rho^2&-4\rho&2(1-\rho^2)\end{bmatrix} \tag{ix}$$

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$$\boldsymbol{\rho}_3{}^{-1} = \frac{1}{2\left(1-3\rho^2\right)}\begin{bmatrix} 1-2\rho^2 & -\rho & 2\rho^2 \\ -\rho & 1 & -2\rho \\ 2\rho^2 & -2\rho & 1-\rho^2 \end{bmatrix} \tag{x}$$

$$\mathbf{J}_3{}^T\boldsymbol{\rho}_3{}^{-1}\mathbf{J}_3 = \frac{5-8\rho-\rho^2}{1-3\rho^2} \tag{xi}$$

$$\boldsymbol{\rho}_3{}^{-1}\mathbf{J}_3 = \frac{1}{1-3\rho^2}\begin{bmatrix} 1-\rho-\rho^2 \\ 1-2\rho \\ 1-2\rho+\rho^2 \end{bmatrix} \tag{xii}$$

$$\boldsymbol{\alpha}_{20} = \frac{\boldsymbol{\rho}_2{}^{-1}\mathbf{J}_2}{\mathbf{J}_2{}^T\boldsymbol{\rho}_2{}^{-1}\mathbf{J}_2} = \frac{1}{5-8\rho-\rho^2}\begin{bmatrix} 1-\rho-\rho^2 \\ 1-2\rho \\ 1-2\rho+\rho^2 \end{bmatrix} \tag{xiii}$$

$$\Rightarrow \qquad \left.\begin{aligned} \alpha_1 = \alpha_5 &= \frac{1-\rho-\rho^2}{5-8\rho-\rho^2} \\ \alpha_2 = \alpha_4 &= \frac{1-2\rho}{5-8\rho-\rho^2} \\ \alpha_3 &= \frac{1-2\rho+\rho^2}{5-8\rho-\rho^2} \end{aligned}\right\} \tag{xiv}$$

Conflict of Interest

The authors declare that there is no conflict of interest.