POISSON BAGUI-LIU-ZHANG DISTRIBUTION: A FLEXIBLE MIXED-POISSON MODEL FOR HEAVY-TAILED COUNT DATA

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Abstract
Some over-dispersed count poses varying characteristics such as uni-modal or multi-modal, right or left skewed, platokurtic or leptokurtic and therefore requires more flexible discrete distributions than the existing ones in order to minimize estimation error. A new discrete distribution named Poisson Bagui-Liu-Zhang distribution for modelling over-dispersed count data has been proposed and its properties such as - hazard function, probability generating function, characteristic function, *rth* factorial moment, raw and central moments, the dispersion index, the coefficient of variation, the coefficient of skewness and the coefficient kurtosis - derived. Conventional estimation methods were used to obtain the estimators for the parameter of the distribution and their performances compared using simulated data. The results from the simulation study showed that the maximum likelihood estimator and the method of proportion estimator of the distribution, have positive bias and showed consistency property while the method of moment estimator and weighted least squares estimator were not consistent and were negatively biased. Generally, the maximum likelihood estimator of the new distribution performed better than the other estimators obtained. Again the new distribution was fitted to two real life data sets and its performance compared to that of the Poisson-Lindley distribution, the Poisson-Akash distribution, the Poisson-Bilal distribution, the geometric distribution and the negative binomial distribution. The results from the data sets, with features; dispersion indices (419.13, 3.527), positive skeweness (3.50, 3.44) and leptokurtic (15.38, 15.69), showed that the new distribution, having produced the minimum values of Akaike information criterion (565.6158, 401.7164), Bayesian information criterion (567.4000,404.4259), negative loglikelihood (281.8079,199.8582) and the highest values Klomogrov-Smirnov/p-value (0.1651/0.3200,0.0942/0.1351) respectively for the two data sets, performed better than the other distributions.
Keywords: Count data, over-dispersion, positive skewness, leptokurtic and mixed-Poisson distribution

1. Introduction

Count data consist of integer values and are obtained by taking counts of events or the number of occurrences of real-life phenomena. These observations are directly generated in several areas of human endeavour such as medicine (the number of cases of a given disease recorded in a hospital), business (the quantity of a given item sold in a grocery store), agriculture (the number of fish mortality recorded in a pond), demography (number of children giving birth to by couples), tourism (the number of immigrants that visit a tourist centre in country) just to mention a few.

Count data are modelled using discrete distributions, especially the Poisson, Bernoulli, binomial, geometric, negative binomial distributions etc. Poisson distribution adequately describes count data when the variance to mean ratio (dispersion index) is one, leading to the concept of equi-dispersion (Johnson *et al.* 2005). However, in several practical situations, we often encounter counts whose dispersion index is greater than one. Such data are called over-dispersed count data. Less frequently also encountered are counts whose dispersion index is less than one and thus are referred to as under-dispersed. Since the Poisson distribution has a

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dispersion index of one, it may not provide a good fit for over-dispersed and under-dispersed count data. Consequently, an over-dispersed discrete distribution is required for over-dispersed count data while an under-dispersed discrete distribution is needed for under-dispersed count data.

Three common methods of deriving discrete distributions for modelling over-dispersion or under-dispersion in count data are the mixed-Poisson method, the weighted Poisson method and the survival discretization method (Eliwa *et al.*, 2020). The advantage of using mixed-Poisson method over the other methods is that it is easy to obtain using the method some mixed-Poisson distributions which are very flexible and simple. Mixed-Poisson method can produce a wide range of flexible mixed-Poisson distributions for modelling over-dispersed count data. Mixed-Poisson distributions are basically distributions obtained by allowing the parameter of the Poisson distribution to follow a given continuous distribution (Raghavachari *et al.*, 1997; Karlis and Xekalaki, 2005; Panjer, 2006). The negative binomial distribution is one amongst the foremost mixed-Poisson distributions. The distribution was obtained by assuming that the parameter of the Poisson distribution is distributed as gamma (Greenwood and Yule, 1920). Although the negative binomial distribution provides good fit for several over-dispersed count data, it may not appropriately model all kinds of over-dispersed count data (Kishore *et al.*, 2018). Some over-dispersed count data may be uni-modal or multi-modal, right or left skewed, platokurtic or leptokurtic and therefore requires more flexible discrete distributions. Some of the mixed-Poisson distributions in literature proposed in order to reduce estimation error include; Poisson-Lindley distribution (Sankaran, 1970), a Poisson mixture of Lindley distribution of Lindley (1958), Poisson- Akash distribution (Shanker, 2017), a Poisson mixture of Akash distribution of Shanker (2015), Poisson-Gold distribution (Hanandeh and Al-Nasser, 2021), a Poisson mixture of Gold distribution of Al-Talib *et al.* (2019), Poisson-Xgamma distribution (Emrah *et al.*, 2022), a Poisson mixture of Xgamma distribution of Sen *et al.* (2016), Poisson Quasi-XLindley distribution (Alghamdi, *et al.*, 2024), a Poiosson mixture of Quasi-XLindley distribution (Ibrahim *et al.*, 2023) and Poisson Komal distribution (Alomaira and Ahsan-ul-Haq, 2025), a Poisson mixture of Shanker (2023).

In order to further increase precision in the modelling of over-dispersed continuous data, Bagui *et al.* (2020) derived the probability density function (pdf) of the exponential mixture of the shifted exponential distribution using the moment generating function approach and named it Bagui-Liu-Zhang distribution. Shortly after, the properties of this distribution such as cumulative distribution function (cdf), moments, coefficients of skewness and kurtosis, reliability function and hazard rate function were obtained by Okereke *et al.* (2022). To the best of our knowledge the Poisson mixture of Bagui-Liu-Zhang distribution has not been derived. Therefore, the aim of this study is to derive a Poisson mixture of Bagui-Liu-Zhang distribution and referred to it as Poisson Bagui-Liu-Zhang distribution (PBLZD), as well as, obtained its properties. The motivation of this study is anchored on the fact that (1) Bagui-Liu-Zhang distribution has an increasing hazard rate function (Okereke *et al.*, 2022), (2) the properties are simple and tractable which makes it easier for application and (3) there is always room for improvement in the modelling of count data, thus we felt that a Poisson mixture of Bagui-Liu-Zhang distribution will have an increasing hazard rate and thus provides greater flexibility in modelling of over-dispersed data.

The rest of the paper is structured as follows; Section 2. consists of the construction of a proposed PBLZD, the generating functions and the moments of the PBLZD are derived in Section 3, estimators of the parameter of the distribution were obtained in Section 4, Section 5 looked at the performance of the distribution using simulated and real-life data and the paper was concluded in Section 6.

2. The Proposed Poisson Bagui-Liu-Zhang Distribution

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This section contains the derivation of a new discrete distribution using a mixed-Poisson approach.

Definition 1: A random variable X follows a Poisson distribution (PD) with parameter λ if the probability mass function (pmf) is given by

$$P(X = x|\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0,1,2,.. \tag{1}$$

Let the probability density function (pdf) of a continuous random variable X with support $(0,\infty)$ be given as $f(x|\delta)$, where δ is the parameter of the pdf. Then the general form of the mixed-Poisson distribution (MPD) is given by

$$P(X = x|\delta) = \int_0^\infty P(X = x|\lambda) f(\lambda|\delta) d\lambda \tag{2}$$

Definition 2: Bagui *et al*. (2020). A random variable X follows a Bagui-Liu-Zhang Distribution (BLZD) with parameter θ if the pdf is given by

$$f_{BLZ(X)}(X = x|\theta) = \theta(1+\theta)e^{-\theta x}\left(1 - e^{-x}\right), \qquad x > 0; \theta > 0 \tag{3}$$

Applying Eq. (2), given Eq. (1) and Eq. (3), we derive the PBLZD for over-dispersed count data as a Poisson mixture of Bagui-Liu-Zhang Distribution.

Proposition 1: A random variable X follows Poisson Bagui-Liu-Zhang Distribution (PBLZD) with parameter θ if the pdf and cumulative density function (cdf) are respectively given by

$$P_{PBLZ}(X = x|\theta) = \theta(1+\theta)\left[\frac{1}{(1+\theta)^{x+1}} - \frac{1}{(2+\theta)^{x+1}}\right], \quad x = 0,1,2,..;\theta > 0 \tag{4}$$

and

$$F_{PBLZ}(X) = P_{PBLZ}(X \le x) = 1 - \left[\frac{1+\theta}{(1+\theta)^{x+1}} - \frac{\theta}{(2+\theta)^{x+1}}\right], \quad x = 0,1,2,..;\theta > 0 \tag{5}$$

Proof:

Let the parameter λ of the pmf in Eq. (1), be a random variable that follows BLZD with parameter θ. Then, by Eq. (3), the pdf of λ is given by

$$f_{BLZ(\lambda)}(\lambda|\theta) = \theta(1+\theta)e^{-\theta\lambda}\left(1 - e^{-\lambda}\right), \qquad \lambda > 0; \theta > 0. \tag{6}$$

Then, by Eq. (2), the pmf of a Poisson Bagui-Liu-Zhang (PBLZ) random variable X can be derived as

$$\begin{aligned} P_{PBLZ}(X = x|\theta) &= \int_0^\infty P(X = x|\lambda) f_{BLZ(\lambda)}(\lambda|\theta) \\ &= \int_0^\infty \frac{\lambda^x e^{-\lambda}}{x!}\theta(1+\theta)e^{-\theta\lambda}\left(1 - e^{-\lambda}\right)d\lambda \\ &= \theta(1+\theta)\int_0^\infty \frac{\lambda^x e^{-\lambda}}{x!} e^{-\theta\lambda}\left(1 - e^{-\lambda}\right)d\lambda \\ &= \theta(1+\theta)\left[\frac{1}{(1+\theta)^{x+1}} - \frac{1}{(2+\theta)^{x+1}}\right], \quad x = 0,1,2,...;\theta > 0. \end{aligned}$$

The cdf is given as

$$F_{PBLZ}(X) = P_{PBLZ}(X \le x) = \sum_{t=0}^{x} P(T = t|\theta)$$

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$$= \sum_{t=0}^{x} \theta(1+\theta)\left[\frac{1}{(1+\theta)^{t+1}} - \frac{1}{(2+\theta)^{t+1}}\right]$$

$$= \theta(1+\theta)\left[\sum_{t=0}^{x}\frac{1}{(1+\theta)^{t+1}} - \sum_{t=0}^{x}\frac{1}{(2+\theta)^{t+1}}\right]$$

$$= \theta(1+\theta)\left[\frac{1}{\theta}\left(1-\frac{1}{(1+\theta)^{x+1}}\right) - \frac{1}{1+\theta}\left(1-\frac{1}{(2+\theta)^{x+1}}\right)\right]$$

$$= 1-\left[\frac{1+\theta}{(1+\theta)^{x+1}} - \frac{\theta}{(2+\theta)^{x+1}}\right], \quad x = 0,1,2,\ldots;\theta > 0$$

The pmf of PBLZD in Eq. (4) is a valid pmf since it satisfies the following properties;

i. $P_{PBLZ}\left(X = x|\theta\right) \geq 0, \forall x = 0,1,2,..$

ii. $\sum_{x=0}^{\infty} P_{PBLZ}\left(X = x|\theta\right) = 1$

Figure 1 shows the plots of the pmf of PBLZD for some values of θ.

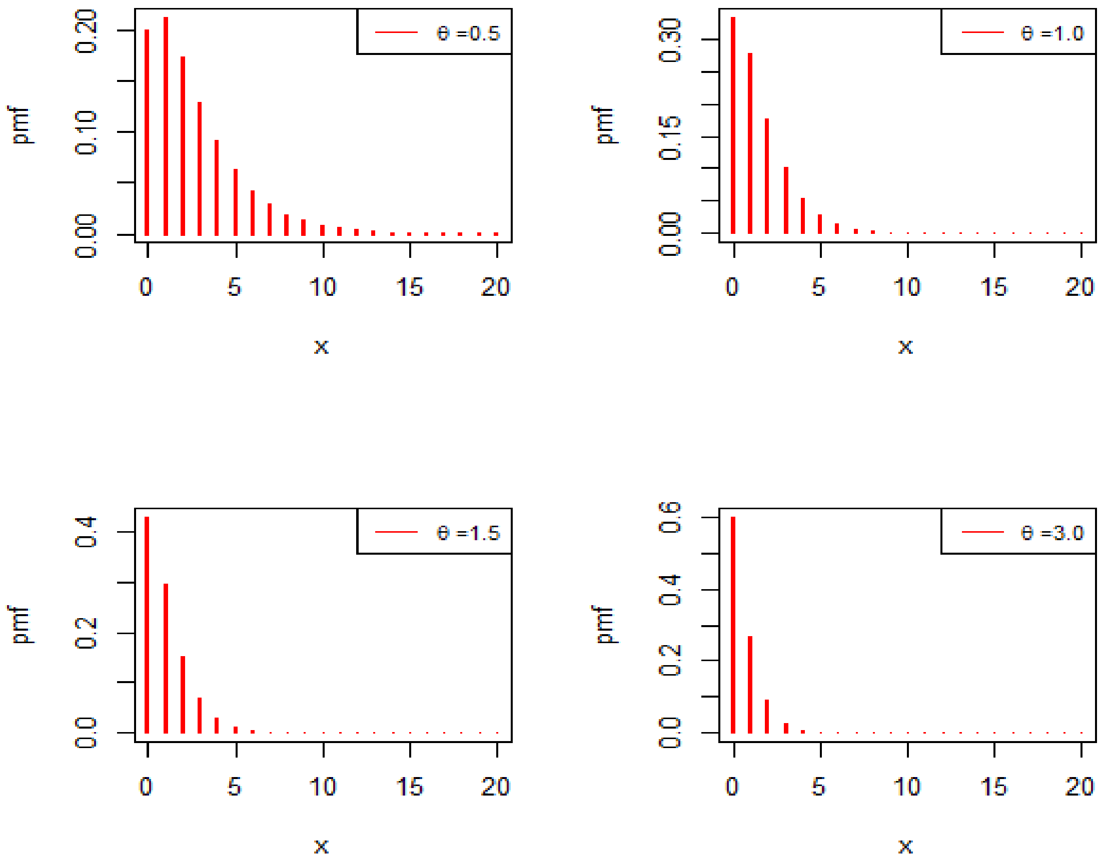

Figure1. The plots of the probability mass function of PBLZD for Some values of θ.

Figure 1 shows that the pmf of PBLZD is a decreasing function for higher values of *X*.

To determine the unimodality of PBLZD or otherwise, we rely on proving its log-concavity. A log-concave distribution, according to Gupta *et al.* (1997), and Bognoli and Bergstrom (2006), is a distribution whose pmf $P(X|\delta)$ conforms to the expression in Eq. (7).

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$$P(X = x+1|\delta)^2 > P(X = x|\delta)P(X = x+2|\delta), \quad x \in \mathrm{N} \tag{7}$$

Eq. (7) also implies

$$\frac{P(X = x+1|\delta)^2}{P(X = x|\delta)P(X = x+2|\delta)} > 1$$

For the pmf of PBLZD in Eq. (4)

$$\frac{P(X = x+1|\theta)^2}{P(X = x|\theta)P(X = x+2|\theta)} =$$

$$\frac{(2+\theta)^{2x+4} + (1+\theta)^{2x+4} - 2(2+3\theta+\theta^2)(2+\theta)^{x+1}(1+\theta)^{x+1}}{(2+\theta)^{2x+4} + (1+\theta)^{2x+4} - \left(2(2+3\theta+\theta^2)(2+\theta)^{x+1}(1+\theta)^{x+1} + (2+\theta)^{x+1}(1+\theta)^{x+1}\right)} \tag{8}$$

Obviously, $$\frac{P(X = x+1|\theta)^2}{P(X = x|\theta)P(X = x+2|\theta)} > 1$$

since $\left(2(2+3\theta+\theta^2)(2+\theta)^{x+1}(1+\theta)^{x+1} + (2+\theta)^{x+1}(1+\theta)^{x+1}\right) > 2(2+3\theta+\theta^2)(2+\theta)^{x+1}(1+\theta)^{x+1}$

Hence the PBLZD is log-concave and consequently unimodal since Log-concavity implies unimodality (Keilson and Gerber, 1971).

The hazard rate function (*hrf*) of a random variable X is given by

$$hrf(x) = \frac{P(x)}{1 - F(x)}$$

where $P(x)$ and $F(x)$ are the probability function and the cumulative function of X respectively

Using Eq. (4) and Eq. (5) the hazard rate function of PBLZD for a random variable X is

$$hrf(x) = \frac{P_{PBLZ}(X = x|\theta)}{1 - F_{PBLZ}(X)} = \theta\left[\frac{(2+\theta)^{x+1} - (1+\theta)^{x+1}}{(2+\theta)^{x+1} - \theta(1+\theta)^{x}}\right], \quad \theta > 0$$

The plots of the *hrf* of PBLZD for some values of θ are shown in Figure 2.

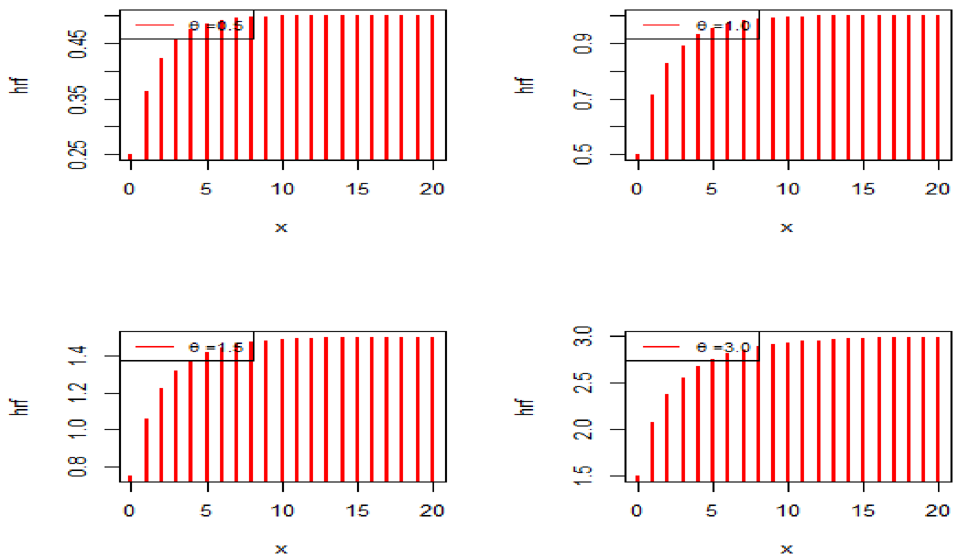

Figure2. The plots of the hazard rate function of PBLZD for Some values of θ.

Figure 2 shows that the PBLZD has an increasing hazard rate function.

3. Generating Functions and Moments of PBLZD

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Here the generating functions of PBLZD such as probability generating function (pgf), and characteristic function (cf) were derived. Also the rth factorial moment of the distribution were obtained and used in obtaining the raw moments and other measures of the distributions.

3.1 Probability Generating Function

Proposition 2: The probability generating function (pgf) of a PBLZ random variable X is given by

$$\rho_X(t)=\theta(1+\theta)\left[\frac{1}{(1+\theta-t)}-\frac{1}{(2+\theta-t)}\right],\quad \theta>0 \tag{9}$$

Proof:

Recall that for every discrete random variable X the pgf is expressed as

$$\rho_X(t)=E\left(t^x\right)=\sum_{\forall x}t^xP(X=x).$$

Thus the pgf of PBZL random variable X is given as

$$\rho_X(t)=\sum_{x=0}^{\infty}t^x\theta(1+\theta)\left[\frac{1}{(1+\theta)^{x+1}}-\frac{1}{(2+\theta)^{x+1}}\right]$$

$$=\theta(1+\theta)\left[\frac{1}{(1+\theta)}\sum_{x=0}^{\infty}\left(\frac{t}{1+\theta}\right)^x-\frac{1}{(2+\theta)}\sum_{x=0}^{\infty}\left(\frac{t}{2+\theta}\right)\right]$$

$$=\theta(1+\theta)\left[\frac{1}{(1+\theta)}\left(\frac{1+\theta-t}{1+\theta}\right)^{-1}-\frac{1}{(2+\theta)}\left(\frac{2+\theta-t}{2+\theta}\right)^{-1}\right]$$

$$=\theta(1+\theta)\left[\frac{1}{(1+\theta-t)}-\frac{1}{(2+\theta-t)}\right],\quad \theta>0$$

3.2 Characteristic Function

Proposition 3: The characteristic function (cf) of a PBLZ random variable X is given by

$$\phi_X(t)=\theta(1+\theta)\left[\frac{1}{\left(1+\theta-e^{it}\right)}-\frac{1}{\left(2+\theta-e^{it}\right)}\right],\quad \theta>0 \tag{10}$$

Proof:

For any random variable X the characteristic function

$$\phi_X(t)=E\left(e^{itx}\right)=\sum_{\forall x}e^{itx}P(X=x)$$

Therefore, for a PBLZ random variable X the cf is obtained as

$$\phi_X(t)=\sum_{x=0}^{\infty}e^{itx}\theta(1+\theta)\left[\frac{1}{(1+\theta)^{x+1}}-\frac{1}{(2+\theta)^{x+1}}\right]$$

$$=\theta(1+\theta)\left[\frac{1}{(1+\theta)}\sum_{x=0}^{\infty}\left(\frac{e^{it}}{1+\theta}\right)^x-\frac{1}{(2+\theta)}\sum_{x=0}^{\infty}\left(\frac{e^{it}}{}\right)^x\right]$$

$$=\theta(1+\theta)\left[\frac{1}{(1+\theta)}\left(\frac{1+\theta-e^{it}}{1+\theta}\right)^{-1}-\frac{1}{(2+\theta)}\left(\frac{2+\theta-e^{it}}{2+\theta}\right)^{-1}\right]$$

$$=\theta(1+\theta)\left[\frac{1}{\left(1+\theta-e^{it}\right)}-\frac{1}{\left(2+\theta-e^{it}\right)}\right],\quad \theta>0$$

3.3 Derivation of Factorial Moments

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Proposition 4: The rth factorial function of a PBLZ random variable X is given by

$$E[(X)_r]_{PBLZ} = \theta(1+\theta)r!\left[\frac{1}{\theta^{r+1}} - \frac{1}{(1+\theta)^{r+1}}\right], \quad r = 1,2,... \tag{11}$$

Proof:
Given the rth factorial moment of a Poisson random variable X as

$$E[(X)_r]_p = m_r(X|\lambda) = \lambda^r,$$

the rth factorial moment of PBLZ random variable X can be derive the as

$$E[(X)_r]_{PBLZ} = E[m_r(X|\lambda)] = E[\lambda^r] = \int_{\forall\mu} \lambda^r f_{BLZ(\lambda)}(\lambda|\theta)d\lambda$$

$$= \int_0^\infty \lambda^r \theta(1+\theta)e^{-\theta\lambda}\left(1-e^{-\lambda}\right)d\lambda$$

$$= \theta(1+\theta)\left[\int_0^\infty \lambda^r e^{-\theta\lambda}d\lambda - \int_0^\infty \lambda^r e^{-(1+\theta)\lambda}d\lambda\right]$$

$$= \theta(1+\theta)r!\left[\frac{1}{\theta^{r+1}} - \frac{1}{(1+\theta)^{r+1}}\right], \quad r = 1,2,..$$

3.4 Moments and Others Measures of PBLZD

Some factorial moments of several distributions can be used to determine the mean, variance, the coefficient of variation, the coefficient of skewness and the coefficient of kurtosis. The general form of the rth factorial moment given by

$$f_r = E[(X)_r] = E(X!) = E[X(X-1)(X-2)(X-3)....(X-(r-1)!)].$$

Representing the rth raw moments as $m_r = \left(E\left(X^r\right)\right)$ and the rth central moment as $\mu_r = \left(E(X-\mu)^r\right)$, the first four raw moments in terms of the factorial moments are given by (Okorie, 2020) as

$$m_1 = f_1 = \mu$$
$$m_2 = f_1 + f_2$$
$$m_3 = f_1 + 3f_2 + f_3$$
$$m_4 = f_1 + 7f_2 + 6f_3 + f_4.$$

Using Eq. (11) and the above relationships, the first four raw moments of a PBLZ random variable X are obtained as;

$$m_1 = E(X) = \frac{1+2\theta}{\theta(1+\theta)} = \text{mean},$$

$$m_2 = E\left(X^2\right) = \frac{2+7\theta+9\theta^2+2\theta^3}{\theta^2(1+\theta)^2},$$

$$m_3 = E\left(X^3\right) = \frac{6+30\theta+61\theta^2+64\theta^3+23\theta^4+2\theta^5}{\theta^3(1+\theta)^3},$$

and

$$m_4 = E\left(X^4\right) = \frac{36+180\theta+536\theta^2+575\theta^3+505\theta^4+279\theta^5+46\theta^6+2\theta^7}{\theta^4(1+\theta)^4}.$$

Again the first four central moments in terms of the raw moments and the factorial moments are given by (Okerie, 2020) are;

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$$\mu_1 = -m_1 + f_1$$
$$\mu_2 = m_1^2 + (1-2m_1)f_1 + f_2$$
$$\mu_3 = -m_1^3 + \left(3m_1^2 - 3m_1 + 1\right)f_1 + 3(1-m_1)f_2 + f_3$$
$$\mu_4 = m_1^4 + \left(-4m_1^3 + 6m_1^2 - 4m_1 + 1\right)f_1 + \left(6m_1^2 - 12m_1 + 7\right)f_2 + (-4m_1 + 6)f_3 + f_4$$

Consequently the first four central moments of a PBLZ random variable X are obtained as;

$$\mu_1 = E(X-\mu) = 0,$$
$$\mu_2 = E(X-\mu)^2 = \frac{1+3\theta+5\theta^2+2\theta^3}{\theta^2(1+\theta)^2} = \text{variance } (Var(X)),$$
$$\mu_3 = E(X-\mu)^3 = \frac{2+9\theta+21\theta^2+40\theta^3+36\theta^4+12\theta^5}{\theta^2(1+\theta)^2},$$

and

$$\mu_4 = E(X-\mu)^4 = \frac{9+6\theta+148\theta^2+443\theta^3+261\theta^4+159\theta^5+33\theta^6+2\theta^7}{\theta^4(1+\theta)^4}.$$

Given the above moments the dispersion index (DI) the coefficient of variation (CV), coefficient of skewness (CS) and coefficient kurtosis (CK) of PBLZD for a discrete random variable X are respectively given as

$$DI = \frac{Var(X)}{E(X)} = \frac{1+3\theta+5\theta^2+2\theta^3}{\theta^2(1+\theta)^2} \cdot \frac{\theta(1+\theta)}{1+2\theta}$$
$$= 1 + \left[\frac{1+2\theta+2\theta^2}{\theta+3\theta^2+2\theta^3}\right],$$

$$CV = \frac{100\sqrt{Var(X)}}{E(X)} = \frac{100\sqrt{1+3\theta+5\theta^2+2\theta^3}}{(1+2\theta)},$$

$$CS = \frac{E(X-\mu)^3}{\left(\sqrt{Var(X)}\right)^3} = \frac{2+9\theta+21\theta^2+40\theta^3+36\theta^4+12\theta^5}{\left(\sqrt{1+3\theta+5\theta^2+2\theta^3}\right)^3} > 0,$$

and

$$CK = \frac{E(X-\mu)^4}{[Var(X)]^2} = \frac{9+6\theta+148\theta^2+443\theta^3+261\theta^4+159\theta^5+33\theta^6+2\theta^7}{\left(1+3\theta+5\theta^2+2\theta^3\right)^2}.$$

Note that $DI > 1$ implying that the PBLZD is over-dispersed and is suitable for modeling over-dispersed count data.

The plots of the mean, variance, DI, CV, CS and CK of PBLZD are given Figure 3.

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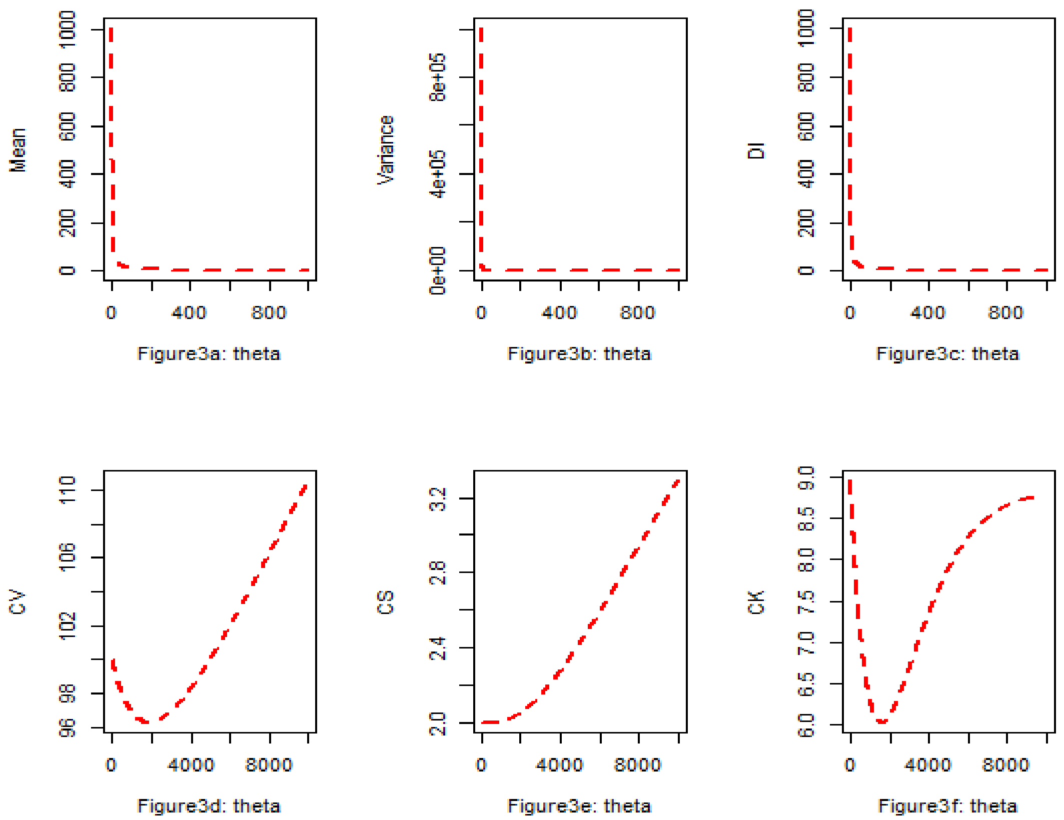

Figure3. The plot of the mean, variance, *DI*, *CV*, *CS* and *CK* of PBLZD

From Figure 3, it can be observed that PBLZD is over-dispersed (Figure 3d), positively skewed (Figure 3e) and leptokurtic with approximately 6.0 as the lower limit of the kurtosis. Additionally, the kurtosis of PBLZD has turning points, a unique feature that implies that the distribution is very flexible for heavy tailed count data.

Furthermore, Table 3.1 shows the numerical values of mean, variance, *DI*, *CS* and *CK* of PBLZD for some selected values of θ.

Table 3.1 Mean, Variance, *DI*, *CS* and *CK* of PBLZD for some Values of θ

	θ								
Measures	**0.1**	**0.3**	**0.5**	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**10.0**
Mean	10.9091	4.1026	2.6667	1.5000	1.0667	0.8333	0.6857	0.5833	0.1909
Variance	111.7355	15.8054	7.1111	2.7500	1.6711	1.1944	0.9274	0.7569	0.2092
DI	10.2424	3.8526	2.6667	1.8333	1.5666	1.4334	1.3524	1.2976	1.0959
CS	2.0061	2.1438	2.4219	3.2892	4.1356	4.9083	5.6144	6.9060	12.5835
CK	6.3191	6.6801	7.8870	8.7686	8.6644	8.5392	8.5044	8.5455	11.2345

It can be observed from Table 3.1 that for the PBLZD:

i. The mean, variance and *DI* vary inversely as θ (also see Figures 3a, 3b and 3c respectively).
ii. The distribution is suitable for modeling over-dispersed count data.
iii. The distribution is positively skewed and the *CS* varies directly as θ.
iv. The distribution is leptokutic.

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Now we examine the minimum and the maximum values of the PBLZD and those of other over-dispersed leptokurtic right skewed discrete distributions such as Poisson-Lindley distribution (PLD), Poisson-Akash distribution (PAD), Geometric distribution (GD), and negative binomial distribution (NBD). The minimum and the maximum values presented in Table 3.2 were obtained by employing calculus through the use of *Symbolab*, an online artificial intelligence (AI) Mathematics solver.

Table 3.2 Minimum and Maximum Values of the Kurtosis of Some Distributions

Pmf	**Kurtosis**	**Etrema** (min , max)
PBLZD	$\dfrac{\left(9+6\theta+148\theta^2+443\theta^3+261\theta^4+159\theta^5+33\theta^6+2\theta^7\right)}{\left(1+3\theta+5\theta^2+2\theta^3\right)^2}$	$(6.03, \infty)$
PLD	$\dfrac{\left(\theta^7+15\theta^6+87\theta^5+258\theta^4+406\theta^3+338\theta^2+144\theta+24\right)}{\left(\theta^3+4\theta^2+6\theta+2\right)^2}$	$(6.0, \infty)$
PAD	$\dfrac{\left(\begin{array}{l}\theta^{11}+10\theta^{10}+30\theta^9+197\theta^8+576\theta^7+120\theta^6+2144\theta^5+2584\theta^4\\+2928\theta^3+2496\theta^2+1440\theta+720\end{array}\right)}{\left(\theta^5+\theta^4+8\theta^3+16\theta^2+12\theta+12\right)^2}$	$(3.8, \infty)$
GD	$6+\dfrac{p^2}{(1-p)}$	(6.00,6.50)
NBD	$\dfrac{6}{r}+\dfrac{p^2}{r(1-p)},\quad r=2$	(3.00,3.25)

From Table 3.2, we can observe that the distributions are leptokurtic. However PBLZD, PLD and PAD have a wider range of value for their kurtosis than GD and NBD.

4. Estimation of the Parameter of PBLZD

The estimators for the parameter of a PBLZD are derived in this Section. Four estimation methods; the maximum likelihood estimation, the method of moment estimation, the method of proportions estimation and the weighted least squares estimation were considered.

4.1 Maximum Likelihood Estimation

The maximum likelihood estimate of the parameter of a probability distribution function is that value of the parameter that maximizes the log-likelihood function of the distribution.

The likelihood function $L(\theta|X)$ and the log-likelihood function $\text{In}L(\theta|X)$ of PBLZ random variable X are respectively given by

$$L(\theta|X)=\underset{i=1}{\overset{n}{\pi}}P(X=x|\theta)=\theta^n(1+\theta)^n\left[(1+\theta)^{-\left(\sum_{i=1}^{n}x_i+n\right)}-(2+\theta)^{-\left(\sum_{i=1}^{n}x_i-n\right)}\right]$$

and

$$\text{In}L(\theta|X)=n\text{In}\theta+n\text{In}(1+\theta)+\text{In}\left[(1+\theta)^{-\left(\sum_{i=1}^{n}x_i+n\right)}-(2+\theta)^{-\left(\sum_{i=1}^{n}x_i+n\right)}\right]. \quad (12)$$

Taking partial derivative of Eq. (12) and equating to zero, we have

$$\frac{\partial \ln L(\theta|X)}{\partial\theta}=\frac{n}{\theta}+\frac{n}{1+\theta}+\left(\sum_{i=1}^{n}x_i+n\right)\left[\frac{(2+\theta)^{-\left(\sum_{i=1}^{n}x_i+n+1\right)}-(1+\theta)^{-\left(\sum_{i=1}^{n}x_i+n+1\right)}}{(1+\theta)^{-\left(\sum_{i=1}^{n}x_i+n\right)}-(2+\theta)^{-\left(\sum_{i=1}^{n}x_i+n\right)}}\right]=0 \qquad (13)$$

Obviously, solving for θ in Eq. (13) will not produce a close form estimator for θ. Thus, the MLE of θ can be obtained numerically using BFGS.

4.2 Method of Moment Estimation

The method of moment estimator (MME) of the parameter of a probability distribution function is obtained by equating theoretical moments with the sample moments. For PBLZD, equating the first theoretical moment with the first sample moment, we have

$$\frac{1+2\theta}{\theta(1+\theta)}=\overline{X}.$$

Solving for θ implies that the method of moment estimator of θ is

$$\hat{\theta}_{MME}=\frac{-(\overline{X}-2)+\sqrt{\overline{X}^2+4}}{2\overline{X}}. \qquad (14)$$

4.3 Method of Proportions Estimation

The method of proportion estimator (MPE), introduced in Khan *et al.* (1989), can be obtained by defining an indicator function

$$\eta(x_i)=\begin{cases}1, & x_i=0\\ 0, & x_i>0\end{cases}$$

such that $\pi_0=\frac{1}{n}\sum_{i=1}^{n}\eta(x_i)$ is the proportion of zeros (0's) contained in the sample.

Since π_0 is an empirical proportion and also consistent with $P(X=0|\theta)$, equating π_0 to $P(X=0|\theta)$ produces the method of proportion estimate for θ.

Equating π_0 to $P(X=0|\theta)$ for the pmf of PBLZD in Eq. (4), we have

$$p_{PBLZ}(X=0|\theta)=\frac{\theta}{2+\theta}=\pi_0$$

Solving for θ, the MPE for θ is

$$\hat{\theta}_{MPE}=\frac{2\pi_0}{1-\pi_0} \qquad (15)$$

4.4 Weighted Least Squares Estimation

Following Afify *et al.* (2021): Let $x_{(1)}<x_{(2)}<.....<x_{(n)}$ be the ordered realization of PBLZD using the cdf in Eq. (5) for $i=1,2,...,n$. Then

$$F_{PBLZ}(x_{(i)}|\theta)=1-\left[\frac{1+\theta}{(1+\theta)^{x_{(i)}+1}}-\frac{\theta}{(2+\theta)^{x_{(i)}+1}}\right].$$

The above theoretical cdf can be estimated using the empirical cdf of PBLZD given by

$$F^{*}_{PBLZ}(x_{(i)}|\theta)=1-\left[\frac{1+\theta}{(1+\theta)^{x_{(i)}+1}}-\frac{\theta}{(2+\theta)^{x_{(i)}+1}}\right],$$

Under weighted least squares method the estimate of θ can be obtained by minimizing the function

$$L_w(\theta)=\sum_{i=1}^{n} W\varepsilon_i^2=\sum_{i=1}^{n} W\left(F_{PBLZ}^{*}\left(x_{(i)}|\theta\right)-F_{PBLZ}\left(x_{(i)}|\theta\right)\right)^2, \tag{16}$$

where the weight W is given by

$$W=\frac{(n+2)(n+1)^2}{i(n-i+1)}.$$

The weighted least squares estimate of θ can be obtained using a numerical methods such as BFGS.

5. Evaluation of the Performance of PBLZD

This Section provides the evaluation of the estimators for the parameter of the PBLZD obtained under the four methods of estimation considered using simulated data. Emphasis is also placed on the applicability of the new distribution to real-world data.

5.1 Simulation Study

A Monte Carlo simulation was conducted 10,000 times for each of the combination of different sample sizes $n=10,25,30,50,100,200$ and different values of the parameter $\theta=0.5,1.0,1.5,2.0,3.0$ of PBLZD, using **Algorithm a**. The simulated data were used to obtain the MLE (using BFGS), MME, MPE and WLSE (using BFGS) for the parameter θ of PBLZD and hence the mean, bias, absolute bias (ABB), means square error (MSE) and mean relative error (MRE) of θ were mathematically given in **Algorithm b**.

Algorithm a. Generation of observations from PBLZD

Step1. Set $r=0$ and specify the value of the parameter θ.

Step2. Generate data from the uniform distribution $\mathrm{U}(0,1)$ and assign it v.

Step3. $r=r+1$

Step4. If $v>\theta(1+\theta)\sum_{i=0}^{r-1}\left[(1+\theta)^{-(i+1)}-(2+\theta)^{-(i+1)}\right]$ goto Step3.

Step5. $x=r-1$

x is a realization of X from the PBLZD.

Algorithm b. Evaluation of the performance of the estimators of the parameter of PBLZD

Step1. Using (Algorithm a) generate $x_1,x_2,\ldots,x_n$ from PBLZD for each combination of samples sizes $n=10,25,30,50,100,200$ and values of the parameter $\theta=0.5,1.0,1.5,2.0,3.0$.

Step2. Compute the MLE, MME, MPE and WLSE of θ for each of the samples

Step3. Conduct Step1 and Step2 10,000 times.

Step4. Calculate the mean, bias, ABB, MSE and MRE respectively as

$$\mathrm{mean}(\theta)=\frac{1}{10000}\sum_{i=1}^{10000}\hat{\theta}_i,\ \mathrm{bias}_\theta(\theta)=\frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{\theta}_i-\theta\right),\ \mathrm{ABB}_\theta(\theta)=\frac{1}{10000}\sum_{i=1}^{10000}\left|\hat{\theta}_i-\theta\right|,$$

$$\mathrm{MSE}_\theta(\theta)=\frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{\theta}_i-\theta\right)^2 \text{ and } \mathrm{MRE}_\theta(\theta)=\frac{1}{10000}\sum_{i=1}^{10000}\frac{\left|\hat{\theta}_i-\theta\right|}{\theta}.$$

The results of the study of the simulated data are shown in Table 5.1-5.5.

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Table 5.1 Numerical results of the MLE, MME, MPE and WLSE for $\theta = 0.5$.

		Estimators			
Measures	n	MLE	MME	MPE	WLSE
mean	10	0.6821	0.1993	0.8341	0.3752
	25	0.6121	0.1960	0.7109	0.3991
	30	0.6001	0.1954	0.6874	0.3988
	50	0.5696	0.1942	0.6422	0.3757
	100	0.5493	0.1936	0.6013	0.3588
	200	0.5340	0.1933	0.5717	0.3467
bias	10	0.0796	-0.3007	0.0718	-0.0233
	25	0.0334	-0.3040	0.0307	-0.0898
	30	0.0246	-0.3046	0.0218	-0.0998
	50	0.0139	-0.3058	0.0131	-0.1239
	100	0.0067	-0.3064	0.0071	-0.1410
	200	0.0034	-0.3067	0.0033	-0.1533
ABB	10	0.1821	0.3007	0.3341	0.1248
	25	0.1121	0.3043	0.2109	0.1009
	30	0.1001	0.3049	0.1874	0.1012
	50	0.0696	0.3058	0.1422	0.1243
	100	0.0493	0.3064	0.1013	0.1411
	200	0.0340	0.3067	0.0717	0.1533
MSE	10	0.0809	0.0917	0.2478	0.0248
	25	0.0233	0.0931	0.0782	0.0150
	30	0.0161	0.0934	0.0514	0.0167
	50	0.0083	0.0937	0.0338	0.0175
	100	0.0039	0.0940	0.0167	0.0208
	200	0.0018	0.0941	0.0081	0.0239
MRE	10	0.3642	0.6014	0.6683	0.2495
	25	0.2121	0.6098	0.4076	0.2059
	30	0.1834	0.6102	0.3211	0.2352
	50	0.1392	0.6115	0.2844	0.2486
	100	0.0987	0.6128	0.2025	0.2821
	200	0.0679	0.6134	0.1433	0.3066

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Table 5.2 Numerical results of the MLE, MME, MPE and WLSE for $\theta = 1.0$.

		Estimators			
Measures	n	MLE	MME	MPE	WLSE
mean	10	1.4332	0.2647	1.6433	0.7246
	25	1.2045	0.2604	1.3896	0.6349
	30	1.1889	0.2598	1.3562	0.6187
	50	1.1570	0.2586	1.2493	0.5826
	100	1.1079	0.2580	1.1702	0.5496
	200	1.0748	0.2576	1.1190	0.5244
bias	10	0.2136	-0.7353	0.2045	-0.2406
	25	0.0654	-0.7396	0.0721	-0.3651
	30	0.0501	-0.7402	0.0511	-0.3813
	50	0.0326	-0.7414	0.0323	-0.4174
	100	0.0165	-.07420	0.0149	-0.4504
	200	0.0075	-0.7424	0.0074	-0.4756
ABB	10	0.4332	0.7353	0.6433	0.2754
	25	0.2045	0.7396	0.3896	0.3651
	30	0.1889	0.7402	0.3562	0.3813
	50	0.1570	0.7414	0.2493	0.4174
	100	0.1079	0.7420	0.1702	0.4504
	200	0.0748	0.7424	0.1190	0.4756
MSE	10	0.5363	0.5424	1.0139	0.1007
	25	0.1124	0.5485	0.2302	0.1462
	30	0.0743	0.5497	0.1832	0.1621
	50	0.0423	0.5500	0.1042	0.1781
	100	0.0192	0.5507	0.0469	0.2045
	200	0.0091	0.5512	0.0227	0.2268
MRE	10	0.4332	0.7353	0.6433	0.2754
	25	0.2100	0.7396	0.3678	0.3823
	30	1.7113	0.7407	0.2994	0.4007
	50	0.1570	0.7414	0.2493	0.4174
	100	0.1079	0.7420	0.1702	0.4504
	200	0.0748	0.7424	0.1190	0.4756

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Table 5.3 Numerical results of the MLE, MME, MPE and WLSE for $\theta = 1.5$.

		Estimators			
Measures	n	MLE	MME	MPE	WLSE
mean	10	2.2161	0.3020	2.4608	0.9358
	25	1.9034	0.2992	2.0169	0.8162
	30	1.8255	0.2988	1.9418	0.7658
	50	1.7546	0.2963	1.8590	0.7200
	100	1.6735	0.2963	1.7452	0.6738
	200	1.6232	0.2960	1.6734	0.6401
bias	10	0.3502	-1.1980	0.3499	-0.5582
	25	0.1098	-1.2007	0.1245	-0.7101
	30	0.0891	-1.2012	0.1023	-0.7342
	50	0.0611	-1.2028	0.0591	-0.7800
	100	0.0263	-1.2037	0.0280	-0.8262
	200	0.0135	-1.2040	0.0116	-0.8599
ABB	10	0.7161	1.1980	0.9608	0.5642
	25	0.4034	1.2007	0.5169	0.6838
	30	0.3255	1.2012	0.4418	0.7342
	50	0.2546	1.2028	0.3590	0.7800
	100	0.1735	1.2037	0.2452	0.8262
	200	0.1232	1.2040	0.1734	0.8599
MSE	10	1.4904	1.4377	2.6226	0.3663
	25	0.3698	1.4396	0.7001	0.4888
	30	0.2562	1.4487	0.5673	0.5879
	50	0.1142	1.4470	0.2203	0.6135
	100	0.0499	1.4492	0.0987	0.6848
	200	0.0246	1.4497	0.0482	0.7405
MRE	10	0.4774	0.7986	0.6405	0.3761
	25	0.2553	0.8007	0.4008	0.4623
	30	0.1463	0.8011	0.3421	0.5010
	50	0.1698	0.8018	0.2393	0.5200
	100	0.1157	0.8025	0.1635	0.5508
	200	0.0822	0.8027	0.1156	0.5733

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Table 5.4 Numerical results of the MLE, MME, MPE and WLSE for $\theta = 2.0$.

		Estimators			
Measures	n	MLE	MME	MPE	WLSE
mean	10	3.0508	0.3311	3.2957	1.0879
	25	2.6104	0.3249	2.7108	0.8759
	30	2.3659	0.3244	2.5621	0.8477
	50	2.4761	0.3239	2.4733	0.8202
	100	2.3659	0.3230	2.3267	0.7653
	200	2.1723	0.3227	2.2279	0.7247
bias	10	0.5624	-1.6689	0.5865	-0.9111
	25	0.1912	-1.6751	0.2011	-1.1241
	30	0.1156	-1.6756	0.0978	-1.1523
	50	0.0917	-1.6761	0.0888	-1.1798
	100	0.0402	-1.6770	0.0399	-1.2347
	200	0.0210	-1.6773	0.0193	-1.2753
ABB	10	1.0508	1.6689	1.2957	0.9121
	25	0.6104	1.6751	0.7108	1.1241
	30	0.4761	1.6756	0.5621	1.1523
	50	0.3659	1.6761	0.4733	1.1798
	100	0.2458	1.6770	0.3267	1.2347
	200	0.1723	1.6773	0.2279	1.2753
MSE	10	2.9881	2.7904	5.5316	0.8920
	25	0.7104	2.8044	1.3521	1.1992
	30	0.4337	2.8057	1.3677	1.2786
	50	0.2363	2.8093	0.3931	1.3981
	100	0.1010	2.8126	0.1785	1.5269
	200	0.0481	2.8133	0.0837	1.6275
MRE	10	0.5254	0.8345	0.6478	0.4561
	25	0.2454	0.8373	0.3233	0.5432
	30	0.2261	0.8378	0.2765	0.5624
	50	0.1829	0.8381	0.2366	0.5899
	100	0.1229	0.8385	0.1633	0.6173
	200	0.0862	0.8386	0.1137	0.6376

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Table 5.5 Numerical results of the MLE, MME, MPE and WLSE of for $\theta = 3.0$.

		Estimators			
Measures	n	MLE	MME	MPE	WLSE
mean	10	4.6690	0.3790	5.0595	1.2855
	25	4.0828	0.3618	4.3254	1.0127
	30	3.9870	0.3601	4.0064	0.9855
	50	3.6129	0.3595	3.7372	0.9622
	100	3.4104	0.3588	3.4960	0.8955
	200	3.2815	0.3584	3.3434	0.8451
bias	10	0.8945	-2.6210	1.0432	-1.7145
	25	0.3496	-2.6382	0.4675	-1.9873
	30	0.2212	-2.6399	0.3141	-2.0145
	50	0.1686	-2.6405	0.1643	-2.0378
	100	0.0774	-2.6412	0.0745	-2.1043
	200	0.0356	-2.6416	0.0374	-2.1549
ABB	10	1.6690	2.6210	2.0595	1.7145
	25	1.0828	2.6382	1.3254	1.9873
	30	0.9870	2.6399	1,0064	2.0145
	50	0.6129	2.6374	0.7372	2.0378
	100	0.4104	2.6412	0.4960	2.1043
	200	0.2815	2.6416	0.3434	2.1549
MSE	10	6.1808	6.9115	13.0814	3.0120
	25	2.3216	6.9704	3.5642	3.7659
	30	1.2044	6.9721	1.8769	3.9734
	50	0.6962	6.9727	1.0026	4.1605
	100	0.2846	6.9763	0.4141	4.4313
	200	0.1296	6.9781	0.1935	4.6453
MRE	10	0.5563	0.8737	0.6855	0.5715
	25	0.2786	0.8796	0.3312	0.6448
	30	0.2342	0.8798	0.2912	0.6634
	50	0.2043	0.8802	0.2457	0.6793
	100	0.1368	0.8804	0.1653	0.7014
	200	0.0938	0.8805	0.1145	0.7183

From Table 5.1-5.5 it can be observed based on the values of the four measures used to do the comparison, that the MLE and the MPE have positive bias and show consistency property, that is, they get better as sample size n increases. The MME and WLSE do not poses consistency property and have negative bias. It can also be observed that the MLE and the MPE performed better at lower values of θ. However the MLE performed better that the other estimators generally.

5.2 Application to Real Data

The performance of PBLZD was examined using real data. Two data sets were used. The first data set, shown in Appendix 1 as given in Afify *et al.* (2021), is the survival time (in days) of 44 patients suffering from head and neck cancer who were treated using a combination of radiotherapy. The second data set presented in Appendix 2 is The Anorexia data (number of

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submissions to animal health from 2003 to 2009), used by Mohammadpour *et al.* (2018). The descriptive statistics of the data sets are presented in Table 5.6.

Table 5.6 Descriptive Statistics of the Data Sets

Data Sets	Mean	Variance	*DI*	*CS*	*CK*
First Data	222.95	93446.42	419.13	3.50	15.38
Second Data	0.82	2.895	3.527	3.442	15.690

Note, from Table 5.6, that the two data sets are positively skewed and over-dispersed. Both data are also leptokurtic.
Having observed the descriptive statistics of the data sets, we then fitted the PBLZD and other discrete distributions for over-dispersed count data such as PLD, PAD, PBD, GD and NBD to the data sets using "fitdistrplus" package in R. The performance of PBLZD and that of the other distributions considered were compared on the basis of Akaike information criterion (AIC), Bayesian information criterion (BIC) and -log-likelihood $(-\ell)$, Chi-Square statistic and Klomogrov-Smirnov (K-S) statistic as shown in the results from the first data set presented in Table 5.7 and that of second data presented in Table 5.8.

Table 5.7 Comparison of Distributions using the First Data Set

$P(x)$	AIC	BIC	$-\ell$	$K-S(p\text{ - value})$
PBLZD	565.6158	567.4000	281.8079	0.1651(0.3200)
PLD	581.0402	582.8243	289.5201	0.3190(0.0200)
PAD	610.9484	612.7326	304.4742	0.3822(0.0000)
PBD	578.8599	580.6441	288.4299	0.2134(0.0424)
GD	566.0102	567.7944	282.0051	0.2476(0.0141)
NBD	567.9999	571.5683	281.9999	0.4102(0.0000)

The results in Table 5.7 show that the PBLZD performed better than other distributions, since it produced minimum values of AIC, BIC, $-\ell$ and $K-S$ with a highest p-value 0.3200 > 0.05.

Table 5.8 Comparison of Distributions using the Second Data Set

$P(x)$	AIC	BIC	$-\ell$	$K-S(p\text{ - value})$
PBLZD	401.7164	404.4259	199.8582	0.0942(0.1351)
PLD	404.4424	406.0800	201.2212	0.1693(0.0721)
PAD	472.4682	474.1058	235.2341	0.2286(0.0043)
PBD	496.2160	497.8536	247.1080	0.3145(0.0012)
GD	598.2576	599.8952	298.1288	0.3211(0.0005)
NBD	884.7561	886.3937	441.3781	0.3400(0.0011)

Table 5.8 shows that the PBLZD again produced minimum values of AIC, BIC, $-\ell$ and K -S with a highest p-value 0.1351 > 0.05. Therefore, the PBLZD performed better than the other distributions.

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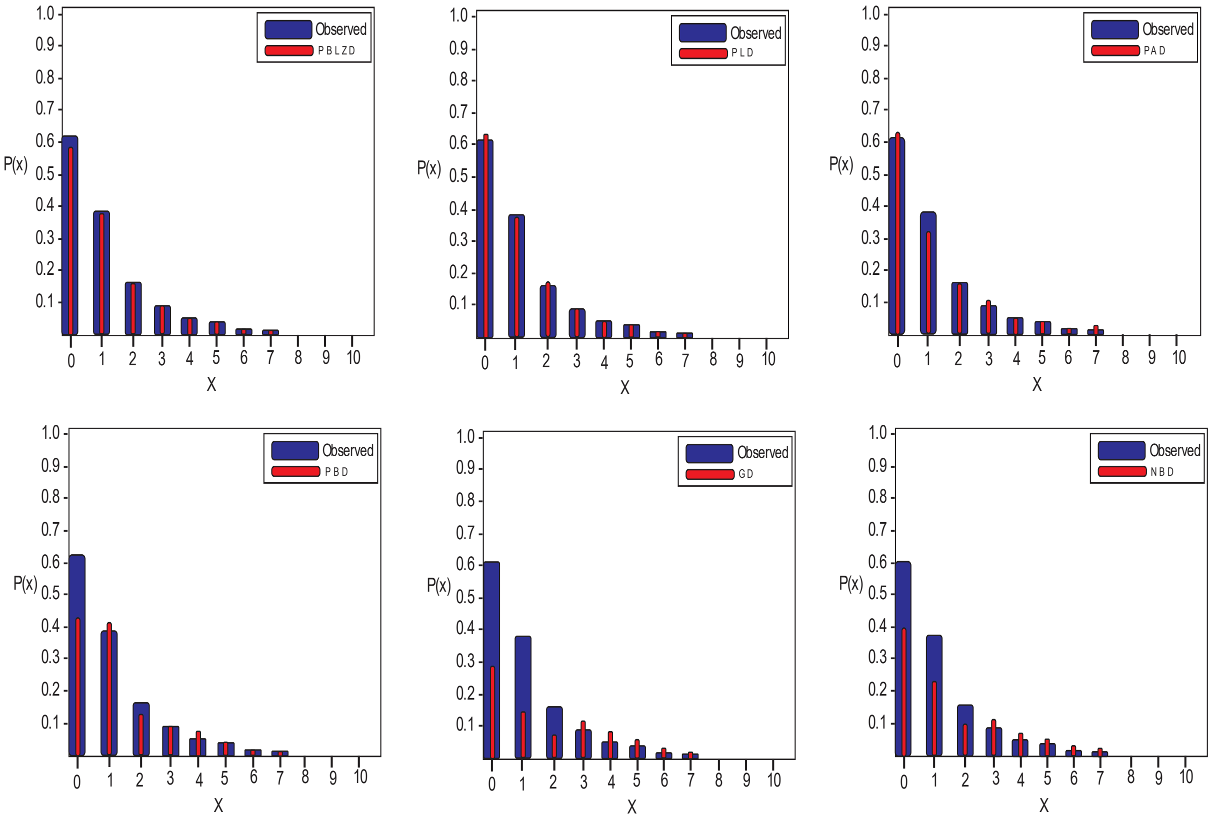

Figure 4. The observed and the fitted Values of the Second Data Set.

6. Conclusion

A one parameter Poisson Bagui-Liu-Zhang distribution (PBLZD) for modelling over-dispersed count data has been proposed. The statistical properties as well as some estimators of the parameter of the new distribution have also been determined. The distribution is unimodal, positively skewed and leptokurtic and will not be appropriate for data set that does not contain these features. A simulation study of the distribution show that the MLE and the MPE of the distribution have positive bias and show consistency property while the MMM and WLSE of the distribution are negatively biased and do not have consistency property. The MLE performed better than the other estimators. An application on two over-dispersed real data set showed that the PBLZD can perform better than some existing discrete distributions for over-dispersed count data most especially when the count data are positively skewed and leptokurtic.

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Appendix 1: Afify *et al.* (2021) . Survival Time (in days) of Patients Suffering from Head and Neck Cancer who are Treated using a Combination of Radiotherapy**.**

12,32, 37, 24, 24, 74, 81, 26, 41, 58, 63, 68, 78, 47, 55, 64, 155, 159, 92, 94, 110, 127, 130, 133, 140, 112, 119, 146, 173, 179, 194, 195, 339, 432, 209, 249, 281, 319, 469, 725, 817, 519, 633, 1776

Appendix 2: The Anorexia data (number of submissions to animal health from 2003 to 2009), used by Mohammadpour *et al.* (2018)
0, 1, 3, 1, 4, 1, 1, 4, 11, 2, 1, 1, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 3, 5, 6, 3, 2, 1, 0, 0, 0, 0, 0, 0, 1, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 4, 0, 1, 0, 1, 0, 0, 0, 2, 1, 0, 0, 0, 0, 0, 0

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