ON THE DEVELOPMENT OF SYSTEM UPDATES FROM SYMMETRIC NONLINEAR STATE-SPACE MODEL

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ABSTRACT:

A new class of nonlinear Time Series model called Symmetric Nonlinear State-Space Model (SNSSM) was developed using Kalman filter technique. Some important components for estimating the SNSSM were successfully derived. These components are estimate of the state, prediction and updating equations which in turn served as system updates of the developed model.

KEY WORDS: State-Space model, Kalman filter, Predicted State, Kalman Gain, Filter State
Covariance

1.0 INTRODUCTION

A State-Space model consists of a transition/state equation and a measurement equation. The transition equation formulates the dynamics of the state variables and the measurement equation relates the observed variables to the unobserved transition vector. The state vector can contain trend, seasonal, cyclical and regression components together with an error term also known as innovation. However, the stochastic behavior of the state variable, its association to the data and the covariance structure of the errors depend on parameters that are almost always unknown,(Sascha, 2009).

The goal of the State-Space model is to infer information about the states, given the observations/measurements, as new information arrives. A well-known algorithm for carrying out this procedure is known as Kalman filter (Anons, 2020). A Kalman filter is an optimal recursive estimator which infers parameters of interest from indirect, inaccurate and uncertain measurements. It is recursive so that new measurements can be processed as they arrive (Lindsay, 2017). Thus, Kalman filter is a set of recursion equations for determining the optimal estimates of the state vector given information available at time t. The filter consist of two sets of equations: prediction equations and updating equations (Shyam*et al.*, 2015). Kalman filter and State-Space model formulation together, provide a very powerful tool for the recursive treatment of dynamic systems, (Amold*et al.*, 2008).

The purpose of filtering is to update our knowledge of the state vector as soon as a new observation becomes available (Raphael, 2016) and (Robert*et al.,* 2016). *Note: in this research, we refer to the original linear State-Space models as Classical State-Space Models (CSSM)*. However, the CSSMs are related to hidden Markov Models; the distinction between the two is that the underlying Markov process (the State) is continuous in the former and discrete in the later.

Researchers from different fields across the world are contributing immensely to the development of the State-Space/Kalman filter models both theoretically and emphatically. Theoretically for example, Hamilton (1994) gave a State-Space representation of a linear dynamic system. The wisdom behind this representation is to capture all the dynamics of the unobserved measurement vector $\mathbf{Y}_t$ in terms of unobserved state vector, *say* $\mathbf{X}_t$. He imposed some restrictions on the parameters of the measurement vector that would ensure the stability of the process. He further proposed a general form of linear State-Space model with a constant parameter; and he derived an optimal forecast of the system via a well-established result for normal variables; all the needed components of the linear Kalman filter algorithm have been derived. The major limitation of the Hamilton's work on the State-Space modeling and Kalman filtering is the assumption of linearity, the frame work was designed to handle a linear or approximately linear system, and therefore, it cannot handle any nonlinear system.

It was observed that the CSSMs had a strong limitation of linearity in its state equation: the model frame work was designed to handle linear or approximately linear systems alone; and majority of real life situations followed nonlinear system! As an improvement, Raphael (2016), proposed a Modified State-Space Model (MSSM). The MSSM allows for the introduction of nonlinear function: Logistic Smooth Transition Autoregressive (LSTAR) model in the state equation of the CSSM and this transformed the CSSM from linear to nonlinear model. The basic limitation of the Raphael's work is the asymmetric behaviour of its state equation as claimed/stated in the Liew (2002) and Olukayode's (2010) arguments.
We seek to address the above limitation by proposing a new class of nonlinear Time Series Model in State-Space form with symmetric nonlinear state equation which is expected to model any symmetric nonlinear series.

2.0 METHODOLOGY

2.1 CLASSICAL STATE-SPACE MODELS (CSSM)

The State-Space model is a system of two equations as given in (1) and (2):

$$Y_{t+1} = HX_{t+1} + V_{t+1} \tag{1}$$

$$X_{t+1} = \psi X_t + \omega_{t+1} \tag{2}$$

The first equation called measurement (observation) equation, describes the relation between the observed Time Series, Y_{t+1} and the (possibly unobserved) state X_{t+1}. The second equation called the (State) transition equation, describes the evolution of the state variables as being driven by the stochastic process of innovations ω_t.

The terms V_{t+1} and ω_{t+1} are the measurement and the process noise respectively. Usually one assumes normal innovations, such that $V_{t+1} \sim N\left(0, \sigma_V^2\right)$ and

$\omega_{t+1} \sim N\left(0, \sigma_{\omega}^{2}\right)$. Similarly, these error terms V_{t+1} and ω_{t+1} are assumed to be serially independent and independent of each other at all time periods as well as uncorrelated with the initial state. The role of V_{t+1} in the output equation (1) is to account for any uncertainty in the measurement of the output (i. e. it tells us how much or little we can trust the equation).

The parameter H is an unknown that links the unobservable variables and regression effects of the state equation with the observation equation, ψ is an unknown parameter that determines how the observation and state equations evolve (change) in time.

Moreover, one can decides to look at (1) and (2) as vectors/matrices which can subsequently be written as

$$\underset{(m\times 1)}{\mathbf{Y}_{t+1}} = \underset{(m\times n)(n\times 1)}{\mathbf{H}\mathbf{X}_{t+1}} + \underset{(m\times 1)}{\mathbf{V}_{t+1}} \tag{3}$$

$$\underset{(n\times 1)}{\mathbf{X}_{t+1}} = \underset{(n\times n)(n\times 1)}{\boldsymbol{\psi}\mathbf{X}_{t}} + \underset{(n\times 1)}{\boldsymbol{\omega}_{t+1}} \tag{4}$$

In this case, $E\left(\mathbf{V}_{t+1}\mathbf{V}'_{t+1}\right) = \underset{m\times m}{\mathbf{L}}$ and $E\left(\boldsymbol{\omega}_{t+1}\boldsymbol{\omega}'_{t+1}\right) = \underset{n\times n}{\mathbf{Z}}$

However, the matrices $\mathbf{V}_{t+1}$ and $\boldsymbol{\omega}_{t+1}$ are not really implemented/included in evaluations of (3) and (4) because they are assumed to be random innovations with zero mean, but instead are always used in determination of any information about the observation and state error covariance matrices $\mathbf{L}$ and $\mathbf{Z}$.

The system matrices $\mathbf{H}$ and $\boldsymbol{\psi}$ are in general vary with time, but would not change with respect to states/transitions. In most cases, they are regarded as constants. $\boldsymbol{\psi}$ Contains the coefficients of the transition terms in the state equation (4) and $\mathbf{H}$ performs similar task in the measurement equation (3).

The above system of equations has a basic limitation in Kalman filter theory as it is linear in parameter. Hence this linearity problem makes it inadequate to handle any nonlinear system; and majority of real life problems in our modern world followed nonlinear system. This necessitates the need for a nonlinear model to overcome this great challenge.

The Extended Kalman Filter (EKF) proposed by Stanley F. Schmidt (1926-2015) and Uncented Kalman Filter (UKF) developed by Jeffrey K. Uhlmann (1965-) were part of the efforts initiated to address the linearity problem of the CSSM. Additionally, Raphael (2016) proposed a modification of the CSSM with the aid of Smooth Transition Autoregressive (STAR) model. This modification is also regarded as a good development in the areas of Time Series as well as Kalman filter literature.

The STAR model is a nonlinear Time Series model that allows for state-dependent or regime-switching behavior. For example, changes in government policy may instigate a change in regime. With a view to modeling this type of Time Series data, a family of Smooth Transition Autoregressive (STAR) models has been proposed by Terasvirta (1994).

The data-generating process to be modeled is viewed as a linear process that switches between numbers of regimes according to some rules. It has been assumed that there is a continuum of switches, that is, there is a smooth transition from one extreme regime to the other. It consists of three stages: specification, estimation and evaluation; Iquebal (2016). The STAR model of order *p* is given as,

$$x_{t+1} = G\left(x_{t+1};\theta\right) = \varphi_1 x_t \left[1 - G\left(X_t;\gamma,c\right)\right] + \varphi_2 x_t G\left(X_t;\gamma,c\right) + \omega_{t+1} \tag{5}$$

where

$G\left(X_t;\gamma,c\right)$ is bounded between 0 and 1, which realizes the "smooth transition" between regimes dynamically rather than an abrupt/sudden jump from one regime to the other. C is the threshold value and the parameter γ determines the speed and smoothness of the transition.

Note that when $\gamma \to \infty$**,** $G\left(X_t;\gamma,c\right) = 1$ then the propose model becomes linear which also happens when $\gamma \to 0$ .Transition Function, $G\left(X_{t-d};\gamma,c\right)$ causes the nonlinear dynamics in the model, and can have different functional choices. For each choice of transition function, we get different regime switching behavior. The most common choices are logistic and exponential forms as given in equation (6) and (7) respectively.

$$G\left(X_{t-d};\gamma,c\right)=\frac{1}{1+\exp[-\gamma\left(X_{t-d}-c\right)]} \tag{6}$$

$$G\left(X_{t-d};\gamma,c\right)=1-\exp[-\gamma\left(X_{t-d}-c\right)^{2}] \tag{7}$$

Note: If (6) is considered as $G\left(X_{t-d};\gamma,c\right)$ in (5), then (5) is called Logistic Smooth Transition Autoregressive (LSTAR) Model. Similarly, if (7) is considered as $G\left(X_{t-d};\gamma,c\right)$ in (5), then (5) is called Exponential Smooth Transition Autoregressive (ESTAR) Model.

Comparing between the two transition functions: (6) and (7), the logistic is changing monotonically with X_t, while the exponential is changing symmetrically at C with X_t. To visualize the asymmetric and symmetric features of the two transition functions: logistic and exponential, see figure 1.

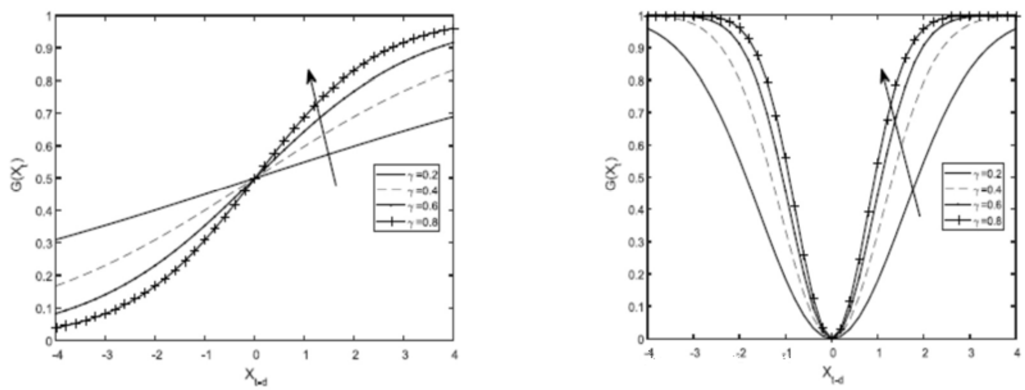

Figure 1: Logistic and Exponential transition functions with varying values of gamma (γ).

3.3 DERIVING THE ESSENTIAL COMPONENTS OF THE PROPOSE SNSSM MODEL

The essential components of the proposed model are Predicted State (PS), and optimal forecast (system update). These would be derive sequentially.

3.3.1 DERIVING THE PREDICTED STATE

Note: equation (7) is known as exponential transition function; if appeared in the STAR model, the result is called an Exponential-STAR (ESTAR) model as in (5) above which is also symmetric in nature. The symmetrical property of equation (5) makes it capable of handling/modeling any symmetric nonlinear series such as exchange rate.

Substituting (7) in (5) we have

$$\begin{aligned} X_{t+1} &= \psi_1 X_t \left(1-\left(1-e^{-\gamma (X_t - C)^2}\right)\right) + \psi_2 X_t \left(1-e^{-\gamma (X_t - C)^2}\right) + \omega_{t+1} \\ &= \psi_1 X_t e^{-\gamma (X_t - C)^2} + \psi_2 X_t \left(1-e^{-\gamma (X_t - C)^2}\right) + \omega_{t+1} \end{aligned} \tag{8}$$

Now, the focus is to get the predicted state estimate which is an important component in the development of the Kalman filter algorithm. To achieve that we differentiate (8) with respect to the current state,

$$\begin{aligned} \frac{\partial X_{t+1}}{\partial X_t} &= \psi_1 X_t \left(-2\left(\gamma\left(X_t - C\right)\right)\right) e^{-\gamma (X_t - C)^2} + \psi_1 e^{-\gamma (X_t - C)^2} \\ &\quad + \psi_2 X_t \left(2\gamma\left(X_t - C\right) e^{-\gamma (X_t - C)^2}\right) + \psi_2 \left(1-e^{-\gamma (X_t - C)^2}\right) \end{aligned} \tag{9}$$

Which gives

$$\begin{aligned} &= -2\psi_1 \gamma X_t \left(X_t - C\right) e^{-\gamma (X_t - C)^2} + \psi_1 e^{-\gamma (X_t - C)^2} + 2\psi_2 \gamma X_t \left(X_t - C\right) e^{-\gamma (X_t - C)^2} \\ &\quad + \psi_2 \left(1-e^{-\gamma (X_t - C)^2}\right) \end{aligned} \tag{10}$$

expanding and equation to zero, we have

$$(\psi_1-\psi_2)\left(1-G(X_t)\right)\left(-2\gamma X_t^2+2\gamma CX_t+1\right)+\psi_2=0 \tag{11}$$

Simplifying further we have

$$-2\gamma\left(X_t-\frac{C}{2}\right)^2=\frac{-\psi_2}{(\psi_1-\psi_2)\left(1-G(X_t)\right)}-\frac{\left(2+\gamma C^2\right)}{2} \tag{12}$$

Taking the L. C. M. of the R. H. S. of (12) and simplifying further gives

$$\hat{X}_t=\frac{-C\pm\sqrt{\dfrac{2\psi_2+\left(\gamma C^2+2\right)(\psi_1-\psi_2)\left(1-G(X_t)\right)}{\gamma(\psi_1-\psi_2)\left(1-G(X_t)\right)}}}{2} \tag{13}$$

Note that (13) is the predicted state estimate with the following regularity conditions:

$C\geq 0,\ 0<\gamma<\infty,\ \psi_1>0,\ \psi_2>0,\ \psi_1>\psi_2,$ and $0<G(X_t)<1.$

Recall that $G(X_t)=G(X_{t-d};\gamma,c)=1-\ell^{-\gamma(X_{t-d}-c)^2}$

One would ask whether the predicted state: equation (13) inherited the symmetrical feature of the exponential transition function given in (7) or not? To answer this, we need to visualize (13) to see if it is really symmetric; even though it will give us two graphs because of the presence of $[\pm]$ signs, the two graphs are given in figures2a and 2b.

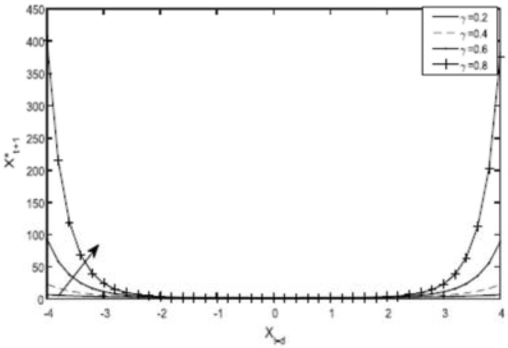

Figure 2a: Estimate of predicted state with varying values of Gamma (γ) for positive sign.

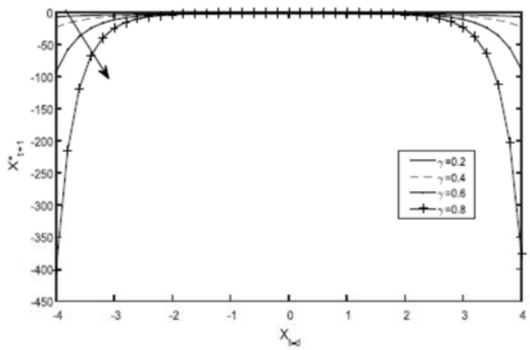

Figure 2b: Estimate of predicted state with varying values of Gamma (γ) for negative sign.

Figure 2a and 2b clearly show that the symmetrical properties of equation (7) were inherited by (13); hence the system of equation: equations: (1) and (2) is now nonlinear as well as symmetric. The system is therefore capable of handling/modeling any symmetric nonlinear series such as exchange rate.

3.4 DERIVATION OF THE OPTIMAL FORECAST

We can get the optimal forecasts from an established marginal and conditional property of the multivariate normal distribution; [see, Rencher (2002), Timm (2002) and Hamilton (1994) for more details]. It should be noted that the easiest way to derive the recursive equations (Kalman recursion) is by using normality assumption. The wisdom behind using the normality assumption is indeed for robustness; that is the solution that may be obtain is optimal in the class of all possible solutions (be it linear and/or nonlinear). We can recall our system of equations (our State-Space equations given in (3) and (4) above) and let $\mathbf{Y}_{t+1}$ and $\mathbf{X}_{t+1}$ denote $(m \times 1)$ and $(n \times 1)$ subvectors respectively whose joint normal distribution is given as

$$\mathbf{J} = \begin{pmatrix} \mathbf{Y}_{t+1} \\ \mathbf{X}_{t+1} \end{pmatrix} \sim \mathbf{N}_{n+r}\left(\begin{pmatrix} \boldsymbol{\mu}_{y} \\ \boldsymbol{\mu}_{x} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\tau}_{yy} & \boldsymbol{\tau}_{yx} \\ \boldsymbol{\tau}_{xy} & \boldsymbol{\tau}_{xx} \end{pmatrix} \right) \quad (14)$$

Similarly, whenever $\mathbf{X}_{t+1}$ becomes available, we use the above normality property to update the distribution of $\mathbf{Y}_{t+1}$.

Hence, the conditional distribution of $\mathbf{Y}_{t+1}$ given $\mathbf{X}_{t+1}$ is also multivariate normal with mean $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\tau}$; i. e. $\mathbf{Y}_{t+1} \mid \mathbf{X}_{t+1} \sim \mathbf{N}(\boldsymbol{\mu}, \boldsymbol{\tau})$; where

$$\boldsymbol{\mu} = \boldsymbol{\mu}_{y} + \boldsymbol{\tau}_{yx}\boldsymbol{\tau}_{xx}^{-1}\left(\mathbf{X}_{t+1} - \boldsymbol{\mu}_{x}\right) \tag{15}$$

$$\boldsymbol{\tau} = \boldsymbol{\tau}_{yy} - \boldsymbol{\tau}_{yx}\boldsymbol{\tau}_{xx}^{-1}\boldsymbol{\tau}_{xy} \tag{16}$$

Now, the optimal forecast value of $\mathbf{Y}_{t+1}$ conditional on having known $\mathbf{X}_{t+1}$ is given as

$$E\left(\mathbf{Y}_{t+1} \mid \mathbf{X}_{t+1}\right) = \boldsymbol{\mu}_{y} + \boldsymbol{\tau}_{yx}\boldsymbol{\tau}_{xx}^{-1}\left(\mathbf{X}_{t+1} - \boldsymbol{\mu}_{x}\right) \tag{17}$$

It is important to point out that the quantity $\boldsymbol{\tau}_{yx}\boldsymbol{\tau}_{xx}^{-1}$ is called a matrix of regression coefficient as it facilitate the role of relating the $E\left(\mathbf{Y}_{t+1} \mid \mathbf{X}_{t+1}\right)$ to $\mathbf{X}_{t+1}$; Rencher, (2002). and the **MSE** of the forecast is given as

$$E\left\{\left(\mathbf{Y}_{t+1} - \boldsymbol{\mu}\right)\left(\mathbf{Y}_{t+1} - \boldsymbol{\mu}\right)' \mid \mathbf{X}_{t+1}\right\} = \boldsymbol{\tau}_{yy} - \boldsymbol{\tau}_{yx}\boldsymbol{\tau}_{xx}^{-1}\boldsymbol{\tau}_{xy} \tag{18}$$

Now, for us to apply equations (15) through (18) in our developed methodology, we need to find $\boldsymbol{\mu}_{y}$, $\boldsymbol{\mu}_{x}$, $\boldsymbol{\tau}_{yy}$, $\boldsymbol{\tau}_{yx}$, $\boldsymbol{\tau}_{xy}$ and $\boldsymbol{\tau}_{xx}$

Now, we can let

$$\boldsymbol{\mu}_{y} = E\left(\mathbf{Y}_{t+1} \mid \mathbf{X}_{t}\right) = \mathbf{H}\hat{\mathbf{X}}_{t+1|t} \tag{19}$$

Since we already know that $E\left(\mathbf{V}_{t+1}\right) = 0$

Considering (3) and (19), we can write the **forecast error** as

$$\left\{\mathbf{Y}_{t+1} - E\left(\mathbf{Y}_{t+1} \mid \mathbf{X}_{t}\right)\right\} = \left(\mathbf{H}\mathbf{X}_{t+1} + \mathbf{V}_{t+1}\right) - \left(\mathbf{H}\hat{\mathbf{X}}_{t+1|t}\right)$$

which resulted to

$$\left\{\mathbf{Y}_{t+1} - E\left(\mathbf{Y}_{t+1} \mid \mathbf{X}_{t}\right)\right\} = \mathbf{H}\left(\mathbf{X}_{t+1} - \hat{\mathbf{X}}_{t+1|t}\right) + \mathbf{V}_{t+1} \tag{20}$$

where $\mathbf{V}_{t+1}$ is as previously defined and it is independent of both $\mathbf{X}_{t+1}$ and $\hat{\mathbf{X}}_{t+1|t}$.

We can now write the **conditional variance** of the forecast error given in (20) as

$$\boldsymbol{\tau}_{yy} = E\left\{\left(\mathbf{Y}_{t+1} - E\left(\mathbf{Y}_{t+1} \mid \mathbf{X}_{t}\right)\right)\left(\mathbf{Y}_{t+1} - E\left(\mathbf{Y}_{t+1} \mid \mathbf{X}_{t}\right)\right)' \mid \mathbf{X}_{t}\right\} \tag{21}$$

by substituting (20) into (21), we have

$$\boldsymbol{\tau}_{yy} = E\left\{\left(\mathbf{H}\left(\mathbf{X}_{t+1} - \hat{\mathbf{X}}_{t+1|t}\right) + \mathbf{V}_{t+1}\right)\left(\mathbf{H}'\left(\mathbf{X}_{t+1} - \hat{\mathbf{X}}_{t+1|t}\right)' + \mathbf{V}'_{t+1}\right)\right\}$$

by expanding and setting all cross products to zero, we have

$$\boldsymbol{\tau}_{yy} = \mathbf{H}E\left(\left(\mathbf{X}_{t+1} - \hat{\mathbf{X}}_{t+1|t}\right)\left(\mathbf{X}_{t+1} - \hat{\mathbf{X}}_{t+1|t}\right)'\right)\mathbf{H}' + E\left(\mathbf{V}_{t+1}\mathbf{V}'_{t+1}\right)$$

which finally gives

$$\boldsymbol{\tau}_{yy} = \mathbf{Q}_{t+1|t} = \mathbf{H}'\mathbf{F}_{t+1|t}\mathbf{H} + \mathbf{L} \tag{22}$$

Furthermore, we can write the **conditional covariance** between the errors in forecasting the observation vector (20) and the state vector as

$$\boldsymbol{\tau}_{yx} = E\left\{\left(\mathbf{Y}_{t+1} - E\left(\mathbf{Y}_{t+1} \mid \mathbf{X}_{t}\right)\right)\left(\mathbf{X}_{t+1} - E\left(\mathbf{X}_{t+1} \mid \mathbf{X}_{t}\right)\right)' \mid \mathbf{X}_{t}\right\} \tag{23}$$

by substituting (20) into (23), we have

$$\boldsymbol{\tau}_{yx} = E\left\{\left(\mathbf{H}\left(\mathbf{X}_{t+1} - \hat{\mathbf{X}}_{t+1|t}\right) + \mathbf{V}_{t+1}\right)\left(\mathbf{X}_{t+1} - \hat{\mathbf{X}}_{t+1|t}\right)' \mid \mathbf{X}_{t}\right\}$$

by expanding and setting all cross products to zero, we have

$$\boldsymbol{\tau}_{yx} = \mathbf{H}E\left(\left(\mathbf{X}_{t+1} - \hat{\mathbf{X}}_{t+1|t}\right)\left(\mathbf{X}_{t+1} - \hat{\mathbf{X}}_{t+1|t}\right)'\right)$$

therefore

$$\boldsymbol{\tau}_{yx} = \mathbf{H}\mathbf{F}_{t+1|t} \tag{24}$$

Now, we can easily write an updated form of $\mathbf{J}$ by substituting the functional forms of $\boldsymbol{\mu}_y$, $\boldsymbol{\mu}_x$, $\boldsymbol{\tau}_{yy}$, $\boldsymbol{\tau}_{yx}$, $\boldsymbol{\tau}_{xy}$ and $\boldsymbol{\tau}_{xx}$ into (14) as

$$\mathbf{J} = \begin{pmatrix} \mathbf{Y}_{t+1} \mid \mathbf{X}_t \\ \mathbf{X}_{t+1} \mid \mathbf{X}_t \end{pmatrix} \sim \mathbf{N}_{n+r}\left(\begin{pmatrix} \mathbf{H}\hat{\mathbf{X}}_{t+1|t} \\ \hat{\mathbf{X}}_{t+1|t} \end{pmatrix}, \begin{pmatrix} \mathbf{H}'\mathbf{F}_{t+1|t}\mathbf{H} + \mathbf{L} & \mathbf{H}\mathbf{F}_{t+1|t} \\ \mathbf{H}'\mathbf{F}_{t+1|t} & \mathbf{F}_{t+1|t} \end{pmatrix}\right) \tag{25}$$

Note that everything about the previous/pass that is needed for the determination of the future values of the observation vector $\mathbf{Y}_{t+1}$ have been summarized and captured by $\mathbf{X}_t$.

With this; we can generalize using the facts from (15) and (16) that $\mathbf{X}_{t+1} \mid \mathbf{X}_{t+1} = \mathbf{X}_{t+1} \mid \mathbf{Y}_{t+1}$ is distributed $\mathbf{N}_r\left(\hat{\mathbf{X}}_{t+1}, \ \mathbf{F}_{t+1}\right)$, where $\hat{\mathbf{X}}_{t+1}$ is given below

$$\hat{\mathbf{X}}_{t+1|t+1} = \hat{\mathbf{X}}_{t+1|t} + \mathbf{H}\mathbf{F}_{t+1|t}\left(\mathbf{H}'\mathbf{F}_{t+1|t}\mathbf{H} + \mathbf{L}\right)^{-1}\left(\mathbf{Y}_{t+1} - \mathbf{H}\hat{\mathbf{X}}_{t+1|t}\right) \tag{26}$$

Note: we can write (26) as

$$\hat{\mathbf{X}}_{t+1|t+1} = \hat{\mathbf{X}}_{t+1|t} + \mathbf{K}_{t+1}\left(\mathbf{Y}_{t+1} - \mathbf{H}\hat{\mathbf{X}}_{t+1|t}\right) \tag{27}$$

Note: Equation (27) is called blending equation; which is obtained by expressing the estimate of the current state $\hat{\mathbf{X}}_{t+1}$ as a linear combination of predicted state $\hat{X}_t$ plus the difference between the actual measurements Y_{t+1} and the predicted state $\hat{X}_t$

multiply by some gain factor called the Kalman gain K_{t+1}. The quantity $\left(\mathbf{Y}_{t+1} - \mathbf{H}\hat{\mathbf{X}}_{t+1|t}\right)$ is called a correction term.

So, the whole idea is if a predicted state is really good, it will be equal to the actual measurement, so the correction term will be zero; and a predicted state will be exactly the estimated state (that is a perfect prediction). On the other hand, if a predicted state is not so good, then the correction term will return a value greater than zero, and the role of the Kalman gain to tellhow much information is needed from the actual measurement to correct a predicted state estimate to get a final more accurate state estimate.

Note: The **MSE** of the forecast given in (18) can now be updated as

$$\mathbf{F}_{t+1|t+1} = \mathbf{F}_{t+1|t} - \mathbf{H}\mathbf{F}_{t+1|t}\left(\mathbf{H}'\mathbf{F}_{t+1|t}\mathbf{H} + \mathbf{L}\right)^{-1}\mathbf{H}'\mathbf{F}_{t+1|t} \tag{28}$$

which can be written (28) as

$$\mathbf{F}_{t+1|t+1} = \mathbf{F}_{t+1|t} - \mathbf{K}_{t+1}\mathbf{H}'\mathbf{F}_{t+1|t} \tag{29}$$

Hence, (29) finally becomes

$$\mathbf{F}_{t+1|t+1} = \left(\mathbf{I} - \mathbf{K}_{t+1}\mathbf{H}'\right)\mathbf{F}_{t+1|t} \tag{30}$$

4.0 FINDINGS AND CONCLUSION

4.1 FINDINGS: KEY EQUATIONS OF THE PROPOSED SNSSM

As stated earlier, the Kalman filter consists of two sets of equations: prediction equations and updating equations. These were derived in the previous section and presented here for crystal clear.

Prediction equations:

$$\underset{(n\times 1)}{\mathbf{X}_{t+1|t}} = \underset{(n\times n)}{\boldsymbol{\psi}}\ \underset{(n\times 1)}{\mathbf{X}_{t|t}}$$
$$\underset{(n\times n)}{\mathbf{F}_{t+1|t}} = \underset{(n\times n)}{\boldsymbol{\psi}}\ \underset{(n\times n)}{\mathbf{F}_{t|t}}\ \underset{(n\times n)}{\boldsymbol{\psi}'} + \underset{(n\times n)}{\mathbf{Z}} \tag{31}$$

where $\mathbf{X}_{t|t}$ is as given in (13)

Updating equations:

$$\underset{(n\times 1)}{\mathbf{X}_{t+1|t+1}} = \underset{(n\times 1)}{\mathbf{X}_{t+1|t}} + \underset{(n\times m)}{\mathbf{K}_{t+1}}\left(\underset{(m\times 1)}{\mathbf{Y}_{t+1}} - \underset{(m\times 1)}{\mathbf{Y}_{t+1|t}}\right)$$
$$\underset{(n\times n)}{\mathbf{F}_{t+1|t+1}} = \left(\underset{(n\times n)}{\mathbf{I}} - \underset{(n\times m)}{\mathbf{K}_{t+1}}\ \underset{(m\times n)}{\mathbf{H}'}\right)\underset{(n\times n)}{\mathbf{F}_{t+1|t}} \tag{32}$$

where $\mathbf{Y}_{t+1|t}$ and $\mathbf{K}_{t+1}$ were given in (19) and (26) respectively.

4.2 CONCLUSION

It is very important to note that (31) and (32) are the system updates developed SNSSM. Running the algorithm at $t=0$ gives one complete Kalman filter's iteration also known as the Kalman recursion. Repeating the same process at $t=1,2,\ldots,T$ (where T is the number of observations) yields the Kalman recursions in Kalman filter literature.

It is customary to initializes/starts the filter with some arbitrary values (a prior information) say $\boldsymbol{\mu}_{0|0}$ and $\mathbf{F}_{0|0}$, use it to predicts $\mathbf{Y}_{1|0}$ and $\mathbf{Q}_{1|0}$; whenever the observation $\mathbf{Y}_1$ becomes available, it will be use in the updating equations and compute $\boldsymbol{\mu}_{1|1}$ and $\mathbf{F}_{1|1}$ which at the same time considered as prior for the subsequent observation. This process completes one Kalman recursion. It is very important to note that the effect of initial prior $\boldsymbol{\mu}_{0|0}$ and $\mathbf{F}_{0|0}$ is decreasing with the increase of time t. This is also consistent with (Yu, 2015), (Tsay, 2010) and (Shyam e*t al.*, 2015).

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