A MODIFIED EXPONENTIAL-TYPE ESTIMATOR FOR POPULATION MEAN WITH TWO AUXILIARY VARIABLES IN TWO PHASE SAMPLING

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ABSTRACT

The ratio estimators of finite population mean are applicable in various fields such as health, education, agriculture etc. Because one of the parameters used by this field in decision making includes the mean and very efficient estimation of this parameter will improve the outcome of such decision. Ratio estimators are applicable when the population mean of auxiliary variable X is known. But in real life situation complete information about population mean are usually unavailable which make ratio estimator impracticable. However, the ratio estimator is always bias and sometimes less efficient. In other to improve the efficiency and reduce the biasedness of existing estimator, the proposed estimators were modified using exponential as one of the improvement strategies. The inability to apply estimators in real-life scenarios stems from a lack of understanding of the auxiliary variable's population mean. The effectiveness of a class of a finite population mean in a double sampling estimator is investigated when the population mean of an auxiliary variable is estimated using exponential-type based on a preliminary large sample. Taylor series expansion up to second degree approximation was used to obtain the suggested estimators' biases and mean square errors. The suggested estimators' efficiency was compared to that of some relevant current estimators in an empirical study using four (4) real life data sets. The suggested estimators have lower mean square errors and higher percentage relative efficiencies than related estimators investigated in the study, according to the results. In addition, the present research can be used in any area of estimation and study variable. The estimators modified in this research work can be used in sampling theory at the estimation stage.

***Keywords*: Estimator, Mean, Efficiency, Mean Square Error, Bias.**

1. Introduction

Auxiliary information has been used in the past to help estimators work more efficiently. To estimate the population mean of the research variable, this information is commonly employed in ratio, product, and regression type estimators. The ratio estimation approach is employed when the correlation between the research variable and the auxiliary variable is positive. If the correlation is negative, on the other hand, the approach of estimating the product is preferable. Some studies, such as those conducted by Searls (1964),Chand (1975), Kiregyera (1984), Murthy (1967), and Singh et al. (2013), used ratio, product, and regression techniques using one or two auxiliary variables.The objective of this study is to estimate a finite population mean for the variable of interest using a class of relative exponential estimators with two auxiliary variables. We looked at a few special estimators from the offered estimators, as well as comparisons between typical multivariate ratio estimators and those proposed by Singh et al (2013). Two variables are addressed with the proposed family of information-based estimators. A lot of work is done to increase the precision of the estimator by including auxiliary data.Furthermore, on the same pattern, Murthy (1967) proposed a product estimator $\overline{y}_p = \overline{y}\frac{\overline{x}}{\overline{X}}$ to estimates population mean $(\overline{y})$. The product estimator is more efficient than the mean per unit whenever estimator $p < \frac{C_x}{2C_y}$

2.Literature Review

The auxiliary information is frequently used to increase precision of the population estimated by taking advantage of the correlation between the study variable and the auxiliary variable. Several authors including Kadilar and Cingi (2006)and Gupta and Shabbir (2010) have proposed different estimators by utilizing information on the auxiliary variable for estimation of population mean. For ratio estimators in sampling theory, population information of the auxiliary variable, such as Coefficient of variable, Coefficient of skewness is often used to increase the efficiency of the estimation for a population mean. Murthy (1964), Chand (1975), Cochran (1977), prasad(1989),Sen (1993), Upadhyaya and singh (1999), Singh and Tailor (2005), singh *et al* (2004), Koyuncu and Kadilar (2009), Lu and Yan (2014),Khan (2016),Olayiwola *et al*. (2020), among others used the population information of the auxiliary variable to increase precision. Though, an attempt had been made by Subramani and Kumarapandiyan (2012), using the linear combination of known population of the auxiliary of known population value of coefficient of Kurtosis and Median of the auxiliary variable, the study is combining the interactive effect of the coefficient of Kurtosis and median of the multiple auxiliary variables to improve the ratio estimators.

Consider a finite population comprises of N units. We draw a sample of size n from the population by using simple random sampling without replacement (SRSWOR). Let y and x be the study and the auxiliary variable of the characteristics y_1 and x_1 respectively for the ith unit. Let $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_1$ and $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_1$ be the sample means and $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_1$ and $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_1$ be the population mean.

Let $s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_1 - \overline{y})^2$ and $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_1 - \overline{x})^2$ be the sample variances.

Also, $S_Y^2 = \frac{1}{N-1}\sum_{i-1}^{N}\left(Y_1 - \bar{Y}\right)^2$ and $S_X^2 = \frac{1}{N-1}\sum_{i-1}^{N}\left(X_1 - \bar{X}\right)^2$ be the population variance. Let p be the correlation coefficient between y and x. Let $c_y = \frac{s_y}{\bar{y}}$ and $c_x = \frac{s_x}{\bar{x}}$ be the coefficient of variation of y and x respectively.

The usual unbiased estimator to estimate the population mean of the study variables is

$$T_0 = \bar{y} \tag{1.1}$$

The variance of the estimator $\bar{y}$ up to first order of approximation is given by

$$Var(T_0) = \bar{y}^2 f_1 C_y^2 \tag{1.2}$$

The usual ratio in two phase sampling and their mean square error are given by

$$T_1 = \frac{\bar{y}}{\bar{x}}\bar{x}' \tag{1.3}$$

$$MES(T_1) = \bar{Y}^2\left[f_1 C_y^2 + f_3\left(C_x^2 - 2\rho_{yx}C_y C_x\right)\right] \tag{1.4}$$

Searls (1964), proposed an estimation procedure for population mean using known knowledge of the coefficient of variation of the auxiliary variable.

$$\bar{y}_1 = a\bar{y} \tag{1.5}$$

$$\text{var}(\bar{y}_1) = (1-B) f_1 \bar{Y}^2 C_y^2 \tag{1.6}$$

Where $a = \{1 + f_1\bar{Y}^2 C_y^2\}$ and $B = f_1\bar{Y}^2 C_y^2$

Chand (1975), proposed the following chain ratio-type estimator in double sampling by incorporating the knowledge of two auxiliary variables, the suggested estimator is given by

$$T_2 = \frac{\bar{y}}{\bar{x}}\frac{\bar{x}'}{\bar{z}'}\bar{z} \tag{1.7}$$

The mean square error of the suggested estimators is given as

$$\text{MSE}\,(T_2) = \bar{y}^2\left[f_1 C_y^2 + f_3\left(C_x^2 - 2p_{yx}C_y C_x\right) + f_2\left(C_z^2 - 2p_{yz}C_y C_z\right)\right] \tag{1.8}$$

Kiregyera (1984), suggested the following chain-type exponential estimators in two phase sampling, the suggested estimators are given as

$$\bar{y}_2 = \frac{\bar{y}}{\bar{x}}\left[\bar{x}' + b_{yx}(\bar{z} - \bar{z}')\right] \tag{1.9}$$

$$\bar{y}_3 = \bar{y} + b_{yx}\left[(\bar{x}' - \bar{x}) - b_{xz}(\bar{z} - \bar{z}')\right] \tag{1.10}$$

The mean square error of the suggested estimators up to first order of approximation are given as following

$$MSE(\bar{y}_2) = \bar{Y}^2\left[f_1C_y^2 + f_3C_x(C_x - 2p_{yx}C_y) + f_2p_{xz}C_x(p_{xz}C_x - 2p_{yx}C_y)\right] \quad (1.11)$$

$$MSE(\bar{y}_3) = \bar{Y}^2C_y^2\left[f_2p_{yx}p_{xz}(p_{yx}p_{xz} - 2p_{yz}) + f_1 - f_3p_{yx}^2\right] \quad (1.12)$$

Singh *et al*. (2013), recommended a class of exponential chain ratio-product type estimator for estimating population mean using two auxiliary variables as follow

$$T_3 = \bar{y}\left[\alpha\exp\left(\frac{\bar{x}'\dfrac{\bar{Z}}{\bar{z}'} - \bar{x}}{\bar{x}'\dfrac{\bar{Z}}{\bar{z}'} + \bar{x}}\right) + \beta\exp\left(\frac{\bar{x} - \bar{x}'\dfrac{\bar{Z}}{\bar{z}'}}{\bar{x} + \bar{x}'\dfrac{\bar{Z}}{\bar{z}'}}\right)\right] \quad (1.13)$$

Where α and β are suitable chosen constants, such that $\alpha + \beta = 1$

The minimum mean square error of the suggested estimator is given as follow

$$MSE(T_3) = \bar{Y}^2C_y^2\left[f_1 - \frac{\left(p_{yx}f_3C_x + p_{yz}f_2C_z\right)^2}{\left(f_3C_x^2 + f_2C_z^2\right)}\right] \quad (1.14)$$

Where the optimum value of α is $\alpha_{opt} = 1/2 + \dfrac{\left(p_{yx}f_3C_{xy} + p_{yz}f_2C_{zy}\right)}{f_3C_x^2 + f_2C_z^2}$

Lu and yan (2014), proposed family of ratio estimators of a class of multivariate ratio estimators using information of two auxiliary variables as follows:

$$\bar{y}_4 = w_1\bar{y}\frac{a_1\bar{X}_1 + b_1}{a_1\bar{x}_1 + b_1} + w_2\bar{y}\frac{a_2\bar{X}_2 + b_2}{a_2\bar{x}_2 + b_2} \quad (1.15)$$

Where w_1 and w_2 are weights that satisfy the condition, such that $w_1 + w_2 = 1$,

MSE of this estimator is given as follows:

$$MSE(\bar{y}_4) = \frac{1-f}{n}\bar{Y}^2\left(\begin{array}{l} C_y^2 + w_1^2a_1^2C_{x_1}^2 + w_2^2a_2^2C_{x_2}^2 - 2w_1a_1p_{yx_1c_yc_{x_1}} \\ -2w_2a_2p_{yx_2c_yc_{x_2}} + 2w_1w_2a_1a_2p_{x_1x_2}C_{x_1}C_{x_2} \end{array}\right) \quad (1.16)$$

Khan (2016), suggested a ratio chain-type exponential estimator for finite population mean of the study variable y, given by

$$T_4 = \bar{y}\left(\frac{\bar{x}' - \bar{x}}{\bar{x}' + \bar{x}}\right)^{k_1} + k_2\left[\bar{x}'\exp\left(\frac{\bar{Z} - \bar{z}'}{\bar{Z} + \bar{z}'}\right) - \bar{x}\right] \quad (1.17)$$

Where k_1 and k_2 are the unknown constants, whose value is to be determined for optimality conditions.

Hence, the mean square error up to first order of approximation, given as

$$MSE(T_4) = \bar{Y}^2 C_y^2 \left[f_1 - f_3 \rho^2{}_{xy} - \frac{f_2 (2\rho_{xy} C_x - \rho_{yz} C_z)^2}{4C_x (C_x - \rho_{xz} C_z) + C_z^2} \right]$$

(1.18)

Olayiwola *et al*. (2020), proposed a class of ratio estimators of a finite population mean using two auxiliary variables under two-phase sample scheme, given as

$$T_5 = \bar{y} \left[a_1 \frac{a_1 \bar{x}' + b_1}{a_1 \bar{x} + b_1} + a_2 \frac{a_2 \bar{x}' + b_2}{a_2 \bar{x} + b_2} \right]$$

(1.19)

Hence, the mean square error up to first order of approximation, given as

$$MSE(T_5)_1 = \bar{Y}^2 \begin{bmatrix} f_1 \left(a_1^2 Q_1^2 C_{x_1}^2 + a_1^2 C_y^2 + a_2^2 Q_2^2 C_{x_2}^2 + a_2^2 C_y^2 + 2a_1 a_2 C_y^2 \right) - f_2 \left(a_1^2 Q_1^2 C_{x_1}^2 + a_2^2 Q_2^2 C_{x_2}^2 \right) \\ -f_3 \begin{pmatrix} 2a_1^2 Q_1 p_{yx_1 c_y c_{x_1}} + 2a_1 a_2 Q_1 p_{yx_1 c_y c_{x_1}} \\ -2a_1 a_2 Q_1 Q_2 p_{x_1 x_2 c_{x_1} c_{x_2}} + 2a_1 a_2 Q_2 p_{yx_2 c_y c_{x_2}} + 2a_2^2 Q_2 p_{yx_2 c_y c_{x_2}} \end{pmatrix} \end{bmatrix}$$

(1.20)

$$MSE(T_5)_{ll} = \bar{Y}^2 \begin{bmatrix} f_1 \begin{pmatrix} a_1^2 Q_1^2 C_{x_1}^2 + a_1^2 Q_2^2 C_{x_2}^2 + a_2^2 C_y^2 - 2a_1^2 Q_1 \rho_{yx_1} C_y C_{x_1} \\ +2a_1 a_2 Q_1 Q_2 \rho_{x_1 x_2} - 2a_1 a_2 Q_1 \rho_{yx_1} C_y C_{x_1} \\ -2a_1 a_2 Q_2 \rho_{yx_2} C_y C_{x_2} + 2a_1 a_2 C_y^2 \\ - 2a_2^2 Q_2 \rho_{yx_2} C_y C_{x_2} \end{pmatrix} \\ + f_2 \left(a_1^2 Q_1^2 C_{x_1}^2 + a_1^2 Q_2^2 C_{x_2}^2 + 2a_1 a_2 Q_1 Q_2 \rho_{x_1 x_2} C_{x_1} C_{x_2} \right) \end{bmatrix}$$

(1.21)

3. Proposed Modified Estimator

Having studied the Olayiwola *et al* (2020), the following estimator for estimating finite population mean is proposed;

$$T_6 = \bar{y} \frac{\bar{x}_1'}{\bar{x}_1} \left[\beta_1 \exp\left(\frac{a_1 \bar{x}_1' + b_1}{a_1 \bar{x}_1 + b_1} \right) + \beta_2 \exp\left(\frac{a_2 \bar{x}_2' + b_2}{a_2 \bar{x}_2 + b_2} \right) \right] \quad (3.1)$$

3.1 Bias and MSE of Estimator T_1

To obtain bias and MSE of T_1, let us define the following;

$$\bar{y}=\left(1+e_0\right),\ \bar{x}_1'=\left(1+e_1'\right)\bar{X}_1,\ \bar{x}_1=\left(1+e_1\right)\bar{X},\ \bar{x}_2=\left(1+e_2\right)\bar{X}_2,\ \bar{x}_2'=\left(1+e_2'\right)\bar{X}_2$$

$$E\left(e_0\right)=E\left(e_1\right)=E\left(e_2\right)=E\left(e_1'\right)=E\left(e_2'\right)=0$$

$$E\left(e_0^2\right)=\theta_1 C_y^2,\ E\left(e_1^2\right)=\theta_1 C_{x_1}^2,\ E\left(e_1'^2\right)=\theta_2 C_{x_1}^2,\ E\left(e_2^2\right)=\theta_1 C_{x_2}^2,\ E\left(e_2'^2\right)=\theta_2 C_{x_2}^2$$

$$E\left(e_0e_1\right)=\theta_1\rho_{yx_1}C_yC_{x_1},\ E\left(e_0e_1'\right)=\theta_2\rho_{yx_1}C_yC_{x_1},\ E\left(e_0e_2\right)=\theta_1\rho_{yx_2}C_yC_{x_2},\ E\left(e_0e_2'\right)=\theta_2\rho_{yx_2}C_yC_{x_2},$$
$$E\left(e_1e_1'\right)=\theta_2 C_{x_1}^2,\ E\left(e_1e_2\right)=\theta_1\rho_{x_1x_2}C_{x_1}C_{x_2},\ E\left(e_1'e_2'\right)=\theta_2\rho_{x_1x_2}C_{x_1}C_{x_2},\ E\left(e_1e_2'\right)=\theta_2\rho_{x_1x_2}C_{x_1}C_{x_2},$$
$$E\left(e_1'e_2\right)=\theta_2\rho_{x_1x_2}C_{x_1}C_{x_2},$$

Re-write (3.1) in error terms, we have;

$$T_1=\bar{Y}\left(1+e_0\right)\frac{\bar{X}_1\left(1+e_1'\right)}{\bar{X}_1\left(1+e_1\right)}\left[\beta_1\exp\left(\frac{a_1\left(1+e_1'\right)\bar{X}_1+b_1}{a_1\left(1+e_1\right)\bar{X}_1+b_1}\right)+\beta_2\exp\left(\frac{a_2\left(1+e_1'\right)\bar{X}_2+b_2}{a_2\left(1+e_1\right)\bar{X}_2+b_2}\right)\right] \tag{3.2}$$

$$=\bar{Y}\left(1+e_0\right)\left(1+e_1'\right)\left(1-e_1+e_1^2\right)\left[\beta_1\exp\left[\left(1+\phi_1e_1'\right)\left(1+\phi_1e_1\right)^{-1}\right]+\beta_2\exp\left[\left(1+\phi_2e_2'\right)\left(1+\phi_2e_2\right)^{-1}\right]\right] \tag{3.3}$$

where $\phi_1=\dfrac{a_1}{\left(a_1+b_1\right)}$

$$T_1=\bar{Y}\left[\begin{array}{l}\beta_1+\beta_2+\beta_1\left[\begin{array}{l}e_0+\left(\frac{4}{5}\phi_1+1\right)e_1'-\left(\frac{4}{5}\phi_1+1\right)e_1+\left(\frac{9}{5}+\phi_1^2\right)e_1^2+\\ \left(\dfrac{\phi_1^2+4\phi_1}{5}\right)e_1'^2+\left(2-\frac{6}{5}\phi_1^2-\frac{8}{5}\phi_1\right)e_1e_1'-\left(\frac{4}{5}\phi_1+1\right)e_0e_1+\left(\frac{4}{5}\phi_1+1\right)e_0e_1'\end{array}\right]\\ +\beta_2\left[\begin{array}{l}e_0-e_1+e_1'-\frac{4}{5}\phi_2e_2+\frac{4}{5}\phi_2e_2'+\phi_2^2e_2^2-\frac{6}{5}\phi_2^2e_2e_2'+\frac{1}{5}\phi_2^2e_2'^2+\frac{4}{5}\phi_2e_2e_1\\ -\frac{4}{5}\phi_2e_1e_2'-\frac{4}{5}\phi_2e_2e_1'+\frac{4}{5}\phi_2e_1'e_2'\end{array}\right]\end{array}\right] \tag{3.4}$$

Subtract $\bar{Y}$ from both sides of (3.4) and taking expectation to obtain bias of T_1 for case I and II as;

$$E(T_1-\bar{Y})=\bar{Y}\left[\begin{array}{l}\beta_1\ \ E\left[\begin{array}{l}\left(\frac{9}{5}+\phi_1^2\right)e_1^2+\left(\dfrac{\phi_1^2+4\phi_1}{5}\right)e_1'^2+\left(2-\frac{6}{5}\phi_1^2-\frac{8}{5}\phi_1\right)e_1e_1'-\left(\frac{4}{5}\phi_1+1\right)e_0e_1\\ +\left(\frac{4}{5}\phi_1+1\right)e_0e_1'\end{array}\right]\\ +\beta_2\ \ E\left[\phi_2^2e_2^2-\frac{6}{5}\phi_2^2e_2e_2'+\frac{1}{5}\phi_2^2e_2'^2+\frac{4}{5}\phi_2e_2e_1-\frac{4}{5}\phi_2e_1e_2'-\frac{4}{5}\phi_2e_2e_1'+\frac{4}{5}\phi_2e_1'e_2'\right]\end{array}\right] \tag{3.5}$$

$$Bias(T_1)_I = \bar{Y}\begin{bmatrix} \beta_1\left[\frac{9}{5}\theta_1 - \frac{4}{5}\theta_2\phi_1 + 2\theta_2 + \phi_1^2\left(\theta_1-\theta_2\right)C_{x_1}^2 + \left(\frac{4}{5}\phi_1+1\right)\left(\theta_2-\theta_1\right)\rho_{yx_1}C_yC_{x_1}\right] \\ +\beta_2\left[\phi_2^2\left(\theta_1-\theta_2\right)C_{x_2}^2 + \frac{4}{5}\phi_2\left(\theta_1-\theta_2\right)\rho_{x_1x_2}C_{x_1}C_{x_2}\right] \end{bmatrix} \quad (3.6)$$

$$Bias(T_1)_{II} = \bar{Y}\begin{bmatrix} \beta_1\left[\left(\frac{9}{5}+\phi_1^2\right)\theta_1 + \left(\frac{\phi_1^2+4\phi_1}{5}\right)\theta_2C_{x_1}^2 - \left(\frac{4}{5}\phi_1+1\right)\rho_{yx_1}C_yC_{x_1}\right] \\ +\beta_2\left[\left(\phi_2^2\theta_1 + \frac{1}{5}\phi_2^2\theta_2\right)C_{x_2}^2 + \frac{4}{5}\phi_2\theta_2\rho_{x_1x_2}C_{x_1}C_{x_2}\right] \end{bmatrix} \quad (3.7)$$

Squaring both sides of (3.5) and taking expectation to obtain MSE of T_1 for case I and II as;

$$MSE\left(T_1\right) = \bar{Y}^2\left[\beta_1\left[e_0 + \left(\frac{4}{5}\phi_1+1\right)e_1' - \left(\frac{4}{5}\phi_1+1\right)e_1\right] + \beta_2\left[e_0 - e_1 + e_1' - \frac{4}{5}\phi_2e_2 + \frac{4}{5}\phi_2e_2'\right]\right]^2 \quad (3.8)$$

$$MSE\left(T_1\right)_I = \bar{Y}^2\begin{bmatrix} \beta_1^2\left[\theta_1C_y^2 + \left(\theta_1-\theta_2\right)\left(\frac{4}{5}\phi_1+1\right)^2C_{x_1}^2 - 2\left(\theta_1-\theta_2\right)\left(\frac{4}{5}\phi_1+1\right)\rho_{yx_1}C_yC_{x_1}\right] \\ +\beta_2^2\begin{bmatrix}\theta_1C_y^2 + \left(\theta_1-\theta_2\right)C_{x_1}^2 + \frac{16}{25}\left(\theta_1-\theta_2\right)\phi_2^2C_{x_2}^2 - 2\left(\theta_1-\theta_2\right)\rho_{yx_1}C_yC_{x_1} \\ -\frac{8}{5}\phi_2\left(\theta_1-\theta_2\right)\rho_{yx_2}C_yC_{x_2}\end{bmatrix} \\ +2\beta_1\beta_2\begin{bmatrix}\theta_1C_y^2 + \left(\frac{4}{5}\phi_1+1\right)\left(\theta_1-\theta_2\right)C_{x_1}^2 - \left(\theta_1-\theta_2\right)\rho_{yx_1}C_yC_{x_1} \\ -\frac{4}{5}\phi_2\left(\theta_1-\theta_2\right)\rho_{yx_2}C_yC_{x_2} - \left(\theta_1-\theta_2\right)\left(\frac{4}{5}\phi_1+1\right)\rho_{yx_1}C_yC_{x_1} \\ +\left(\frac{4}{5}\phi_1+1\right)\frac{4}{5}\phi_2\left(\theta_1-\theta_2\right)_1\rho_{x_1x_2}C_{x_1}C_{x_2}\end{bmatrix} \end{bmatrix} \quad (3.9)$$

$$MSE\left(T_1\right)_{II} = \bar{Y}^2\begin{bmatrix} \beta_1^2\left[\theta_1C_y^2 + \left(\theta_2+\theta_1\right)\left(\frac{4}{5}\phi_1+1\right)^2C_{x_1}^2 - 2\left(\frac{4}{5}\phi_1+1\right)\theta_1\rho_{yx_1}C_yC_{x_1}\right] \\ +\beta_2^2\begin{bmatrix}\theta_1C_y^2 + \left(\theta_2+\theta_1\right)C_{x_1}^2 + \frac{16}{25}\left(\theta_2+\theta_1\right)\phi_2^2C_{x_2}^2 - \theta_1\rho_{yx_1}C_yC_{x_1} - \frac{8}{5}\phi_2\theta_1\rho_{yx_2}C_yC_{x_2} \\ +\frac{8}{5}\phi_2\left(\theta_2+\theta_1\right)\rho_{x_1x_2}C_{x_1}C_{x_2}\end{bmatrix} \\ +2\beta_1\beta_2\begin{bmatrix}\theta_1C_y^2 + \left(\frac{4}{5}\phi_1+1\right)\left(\theta_2+\theta_1\right)C_{x_1}^2 - \left(\frac{4}{5}\phi_1+1\right)\theta_1\rho_{yx_1}C_yC_{x_1} - \frac{4}{5}\phi_2\theta_1\rho_{yx_2}C_yC_{x_2} \\ +\left(\frac{4}{5}\phi_1+1\right)\frac{4}{5}\phi_2\left(\theta_2+\theta_1\right)_1\rho_{x_1x_2}C_{x_1}C_{x_2}\end{bmatrix} \end{bmatrix} \quad (3.10)$$

To obtain minimum mean square error in case I and II, differentiating (3.9) and (3.10) partially w.r.t β_1 and β_2 and equal to zero and therefore, substitute the values of β_1 and β_2 back into (3.9) and (3.10)

$$\frac{\partial MSE\left(T_1\right)_I}{\partial\beta_1} = \bar{Y}^2\left[2A\beta_1 + 2B\beta_2\right] = 0 \quad (3.11)$$

Where

$$A=\theta_1 C_y^2+\left(\theta_1-\theta_2\right)\left(\tfrac{4}{5}\phi_1+1\right)^2 C_{x_1}^2-2\left(\theta_1-\theta_2\right)\left(\tfrac{4}{5}\phi_1+1\right)\rho_{yx_1}C_yC_{x_1},$$

$$B=\theta_1 C_y^2+\left(\tfrac{4}{5}\phi_1+1\right)\left(\theta_1-2\theta_2\right)C_{x_1}^2-\left(\theta_1-\theta_2\right)\rho_{yx_1}C_yC_{x_1}-\tfrac{4}{5}\phi_2\left(\theta_1-\theta_2\right)\rho_{yx_2}C_yC_{x_2}-\left(\theta_1-\theta_2\right)\left(\tfrac{4}{5}\phi_1+1\right)\rho_{yx_1}C_yC_{x_1}$$
$$+\left(\tfrac{4}{5}\phi_1+1\right)\tfrac{4}{5}\phi_2\left(\theta_1-\theta_2\right)_1\rho_{x_1x_2}C_{x_1}C_{x_2},$$

$$C=\theta_1 C_y^2+\left(\theta_1-\theta_2\right)C_{x_1}^2+\tfrac{16}{25}\left(\theta_1-\theta_2\right)\phi_2^2C_{x_2}^2-2\left(\theta_1-\theta_2\right)\rho_{yx_1}C_yC_{x_1}-\tfrac{8}{5}\phi_2\left(\theta_1-\theta_2\right)\rho_{yx_2}C_yC_{x_2}$$

$$\beta_1=\frac{-B\beta_2}{A}\quad\text{but }\beta_2=1-\beta_1\text{ and }\beta_1=\frac{-B}{A-B}$$

$$\frac{\partial MSE\left(T_1\right)_I}{\partial\beta_2}=\bar{Y}^2\left[2\beta_2\begin{bmatrix}\theta_1 C_y^2+\left(\theta_1-\theta_2\right)C_{x_1}^2+\tfrac{16}{25}\left(\theta_1-\theta_2\right)\phi_2^2C_{x_2}^2-2\left(\theta_1-\theta_2\right)\rho_{yx_1}C_yC_{x_1}\\-\tfrac{8}{5}\phi_2\left(\theta_1-\theta_2\right)\rho_{yx_2}C_yC_{x_2}\end{bmatrix}+2\beta_1\left[B\right]\right]=0$$

$$\beta_2=\frac{-B}{B+A}$$

$$MSE\left(T_1\right)_{I\min}=\bar{Y}^2\left[\frac{B^2}{\left(A-B\right)^2}A+\frac{B^2}{\left(C-B\right)^2}C+2\frac{B^3}{\left(A+B\right)\left(B+C\right)}\right]\qquad(3.12)$$

$$\frac{\partial MSE\left(T_1\right)_{11}}{\partial\beta_1}=\bar{Y}^2\left[2\beta_1K+2\beta_2M\right]=0$$

$$\beta_1=\frac{-M}{K-M}$$

$$\frac{\partial MSE\left(T_1\right)_{11}}{\partial\beta_2}=\bar{Y}^2\left[2\beta_2L+2\beta_1M\right]=0$$

$$\beta_2=\frac{-M}{L-M}$$

Where

$$K=\theta_1 C_y^2+\left(\tfrac{4}{5}\phi_1+1\right)^2\theta_2C_{x_1}^2+\left(\tfrac{4}{5}\phi_1+1\right)^2\theta_1C_{x_1}^2-2\left(\tfrac{4}{5}\phi_1+1\right)\theta_1\rho_{yx_1}C_yC_{x_1}$$

$$L=\theta_1 C_y^2+\left(\theta_2+\theta_1\right)C_{x_1}^2+\tfrac{16}{25}\left(\theta_2+\theta_1\right)\phi_2^2C_{x_2}^2-\theta_1\rho_{yx_1}C_yC_{x_1}-\tfrac{8}{5}\phi_2\theta_1\rho_{yx_2}C_yC_{x_2}+\tfrac{8}{5}\phi_2\left(\theta_2+\theta_1\right)\rho_{x_1x_2}C_{x_1}C_{x_2}$$

$$M=\left[\theta_1 C_y^2+\left(\tfrac{4}{5}\phi_1+1\right)\left(\theta_2+\theta_1\right)C_{x_1}^2-\left(2+\tfrac{4}{5}\phi_1\right)\theta_1\rho_{yx_1}C_yC_{x_1}-\tfrac{4}{5}\phi_2\theta_1\rho_{yx_2}C_yC_{x_2}+\left(\tfrac{4}{5}\phi_1+1\right)\tfrac{4}{5}\phi_2\left(\theta_2+\theta_1\right)_1\rho_{x_1x_2}C_{x_1}C_{x_2}\right]$$

$$MSE\left(T_1\right)_{II\min}=\bar{Y}^2\left[\frac{M^2}{\left(K-M\right)^2}K+\frac{M^2}{\left(L-M\right)^2}L+2\frac{M^2}{\left(K-M\right)\left(L-M\right)}M\right]\qquad(3.13)$$

4. Empirical Study

To examine the merit of the suggested estimator, we have considered five natural population data sets. The descriptions of the population are given below.

Data 1: Lu (2014) doi:10.1371/journal.pone.0089538.tool

$$N=180; n'=100; n=70; \bar{Y}=13.9951; \bar{X}_1=27.3981; \bar{X}_2=38.7167; C_y=0.4180; C_{x_1}=0.4254;$$
$$C_{x_2}=0.3339; \rho_{yx_1}=0.5630; \rho_{yx_2}=0.5273; \rho_{x_1x_2}=0.2589; \phi_1=0.002; \phi_2=1.6519$$

Table1: Shows the MSE and PRE of Proposed and Existing Estimators Using Data I

ESTIMATORS	MSE	PRE
T_0	0.2987629	100
T_1	0.2825984	105.72
T_2	0.2515201	118.7829
T_3	0.2492032	119.8873
T_4	0.2406898	125.4952
T_{5_I}	0.4510301	66.24013
$T_{5_{II}}$	0.2896271	103.1544
T_{6_I} **(proposed)**	4.234848e-05	705486.8
$T_{6_{II}}$ **(proposed)**	0.07434008	401.8868

Data 2: Lu (2014)

$$N=180; n'=90; n=50; \bar{Y}=13.9951; \bar{X}_1=17.3981; \bar{X}_2=28.7167; C_y=0.4180; C_{x_1}=0.4254;$$
$$C_{x_2}=0.3339; \rho_{yx_1}=0.5630; \rho_{yx_2}=0.5273; \rho_{x_1x_2}=0.2589; \phi_1=0.002; \phi_2=1.6519$$

Table2: Shows the MSE and PRE of Proposed and Existing Estimators Using Data II

ESTIMATORS	MSE	PRE
T_0	0.4943169	100
T_1	0.4607903	107.2759
T_2	0.4219426	117.1526

T_3	0.3989339	123.9095
T_4	0.3787375	130.517
T_{5_I}	0.7376306	67.01415
$T_{5_{II}}$	0.2585298	191.203
T_{6_I} **(proposed)**	0.3725505	563.1502
$T_{6_{II}}$ **(proposed)**	0.09039989	4002.159

Data 3: (Source: Data used by Anderson (1958)); (25 families have been observed for the following three variable.) Y: Head length of second son; X_1: Head length of first son; X_2: Head breadth of first sons

$$N = 25; n' = 10; n = 7; \bar{Y} = 183.84; \bar{X}_1 = 185.72; \bar{X}_2 = 151.12; C_y = 0.0546; C_{x_1} = 0.0526;$$
$$C_{x_2} = 0.0488; \rho_{yx_1} = 0.7326; \rho_{yx_2} = 0.6430; \rho_{x_1x_2} = 0.6837; \phi_1 = 0.002; \phi_2 = 1.6519$$

Table3: Shows the MSE and PRE of Proposed and Existing Estimators Using Data III

ESTIMATORS	MSE	PRE
T_0	10.36334	100
T_1	8.457155	122.5393
T_2	5.795437	178.819
T_3	9.927072	104.3947
T_4	6.465406	160.2891
T_{5_I}	10.97317	94.44252
$T_{5_{II}}$	2.222643	466.2621
T_{6_I} **(proposed)**	5.477815	189.1875
$T_{6_{II}}$ **(proposed)**	10.05151	103.1023

5.1 Conclusion

A class of relative exponential-type estimator for population mean employing two auxiliary variables was suggested under two phase sampling in the study. The proposed estimator's MSE and PRE (T_{61} and T_{62}) were computed, and the improved estimator had the lowest Mean Square Error (MSE). In comparison to other relative estimators, the percentage Relative Efficiency (PRE) is higher. As a result, the proposed estimator outperforms the existing estimators in this investigation.

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