Parameter (shape) Estimation of Weibull-Exponential Distribution Using Classical and Bayansian Approach Under Different Loss Functions

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Abstract

The Bayesian as a statistical approach is a method applied in statistical inference which helps researchers to incorporate prior information surrounding the population parameter with support from information embodied in a sample to guide the inference process. From the Bayesian viewpoint, the choice of prior depends on the one's wide knowledge of the subject matter, since there is no obvious approach from which one can decisively conclude that one prior has edge over the other. This paper aim at studied the parameter (shape) of Weibull-exponential distribution via classical and the Bayesian approach. Different estimates of the parameter (shape) were obtained from the Bayesian approach using quasi and extended Jeffery priors, under various loss functions. The results shows that the quadratic loss functions under extended Jeffrey prior and quasi prior outperformed the squared error loss function and the precautionary loss function across different sample sizes. The result also reveals that the Bayesian estimate of the parameter (shape) under extended Jeffrey and quasi prior using quadratic loss function is better than the maximum likelihood estimate. Finally, it was deduced that, an increment in the sample size, makes the error to reduce and the estimates approach the real value of the parameter (shape).
Keyword: Weibull-exponential distribution, Classical approach, Bayesian method, Quasi prior, extended Jeffrey prior. Posterior distribution.

1. Introduction

In the study of probability and statistics, the weibull-exponential distribution (WED) is one of the form of the generalization of exponential distribution. The generalization of exponential distribution is as a result of its constant failure rate. This feature of the exponential distribution brought about incompatibility of the distribution to model real life problem. The weibull-exponential distribution is more flexible and suitable in modeling some real life problem. Oguntunde *et al* (2015) studied the weibull-distribution, and applied it to real life scenario. The classical method of estimating parameter of any distribution requires no prior information about the parameter to be estimated. The Bayesian approach needs suitable choice of prior information for the parameters. According to Arnold & Press (1983) they opined thatt, from a Bayesian point of view, there is no enough evidence to claim that one prior is better than any other. In presumption, one has one's opinion on the chosen prior and living with all of its lumps and bumps become compulsory. However, if enough information is generated about the parameter(s) then it is reasonable enough to usef the informative prior(s) which is better over other choices. Otherwise vague priors or non informative prior may be considered . This paper aim at using quasi and extended Jeffery prior as against the uniform and Jeffery priors previously used for the same parameter estimation.

2. Literature Review

Series of paper have been published in parameters estimation of exponential distribution. Oguntunde *et al* (2015) estimate the parameters of weibull-exponential distribution using Maximum likelihood method. Recently Ieren and Oguntunde (2018) estimate the parameter (shape) of the weibull exponential distribution based on uniform and Jeffery prior under square error, quadratic and precautionary loss functions. This paper intends to estimate the parameter (shape) of this distribution under extended Jeffery and quasi prior and obtain the estimates as well as the relative posterior risk. Abdalla and Junping (2019) estimate the parameter (shape) and reliability function of one-parameter Burr-X distribution using The expectation of Bayesian estimate method and the maximum likelihood method. Isha Gupta and Rahul Gupta (2018) Investigate the Bayesian and Non-Bayesian estimation methods scale parameter of Gamma distribution. Afaq *et al.* (2015)in their paper, compare the priors for the exponentiated exponential distribution considering different loss functions. Aliyu and Yahaya (2016) studied the estimation of parameter (shape) of Generalized Rayleigh distribution with assumption of non-informative prior under squared error, Entropy and Precautionary loss functions. Abbas and Tang (2015) estimate the Fréchet distribution parameter putting reference priors into consideration. Many authors have made their contributions towards the development of bayesian method, which includes but not limited to Arnold (1983), Terma and Oguntunde (2018), Terna and Angela (2018), *Fatima and Ahmad* (2017), Tahir *et al* (2016). The focus of this paper is to apply the Bayesian method of estimating the parameter of Weibull exponential distribution using quasi and extended jeffery prior under different loss function. The compare the results of the estimators with the maximum likelihood method using mean square error criteria.

3. Materials and Method

This section considered the estimation of the parameter (shape) of weibull-exponential distribution using maximum likelihood approach and Bayesian approach. Under the Bayesian approach, two main prior distributions are considered in estimating the posterior distribution of the parameter (shape); thus extended Jeffery prior and quasi prior. After deriving the posterior distribution, three loss functions, thus the square error loss function, quadratic loss function, precautionary loss function were employed to derived the estimators through which the best estimator is selected using mean square error (MSE) criteria. With this, the estimator with the smallest estimate is considered to be the best estimator of the parameter (shape) of the Weibull exponential distribution. The approach is under listed in the subsection three (3)

3.1 Estimation Method

3.1 Maximum Likelihood Estimation

The pdf is expressed as

$$f\left(x\right)=\alpha\beta\lambda e^{\lambda\beta x}\left(1-e^{-\lambda x}\right)^{\beta-1}e^{-\alpha\left(e^{\lambda x-1}\right)^{\beta}}$$

and the cdf is also expressed

(1)

$$F(x) = 1 - e^{-\alpha\left(e^{\lambda x}-1\right)^{\beta}} \tag{2}$$

where α and β are the parameter (shape)s and λ is the scale parameter of the distribution.

The log likelihood function for α, β, λ is obtained from probability density function as expressed as follow;

$$L(X_1, X_2, ..., X_n / \alpha, \beta, \lambda) = (\alpha\beta\lambda)^n e^{\lambda\beta\sum_{i=1}^{n} x_i} \sum_{i=1}^{n}\left(1-e^{-\lambda x_i}\right)^{\beta-1} \exp\left\{-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right\}$$

and the Maximum likelihood estimator of the parameter (shape) is obtained as

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} \tag{3}$$

3.2. Bayes Estimation Using Extended Jeffrey Prior under Various Loss Function and Its Correspondent Risk

Posterior distribution assuming extended Jeffrey prior

The extended Jeffrey prior is given as $p(\alpha) \propto \frac{1}{\alpha^{2c}}, \quad \alpha > 0$

And the posterior distribution of parameter α is obtained as

$$p(\alpha / \underline{x}) = \frac{\alpha^{n-2c} e^{-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} \left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{n-2c+1}}{\Gamma(n-2c+1)} \tag{5}$$

i. Squared Error Loss Function (SELF) using extended Jeffrey prior.
The squared error loss function associated with the parameter α is defined as;

$$L(\alpha, \alpha_{SELF}) = (\alpha - \alpha_{SELF})^2 \tag{6}$$

Where α_{SELF} is the estimator of the parameter using SELF and it is obtained as;

$$\alpha_{SELF} = \frac{\Gamma(n-2c+2)}{\Gamma(n-2c+1)\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} \tag{7}$$

And the risk is

$$p(\alpha_{SELF}) = \frac{\Gamma(n-2c+1)\Gamma(n-2c+3) - \left(\Gamma(n-2c+1)\right)^2}{\left(\Gamma(n-2c+1)\right)^2 - \left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^2} \tag{8}$$

ii. Quadratic Loss Function Using Extended Jeffrey Prior.
The quadratic loss function (QLF) associated with the parameter α is defined as;

$$L\left(\alpha,\alpha_{QLF}\right)=\left(\frac{\alpha-\alpha_{QLF}}{\alpha}\right)^{2} \tag{9}$$

Where α_{QLF} is the estimator of the parameter using QLF and it is obtained as;

$$\alpha_{QLF}=\frac{\Gamma\left(n-2c\right)}{\Gamma\left(n-2c-1\right)\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} \tag{10}$$

and the risk is

$$p\left(\alpha_{QLF}\right)=\frac{\Gamma\left(n-2c+1\right)\Gamma\left(n-2c-1\right)-\Gamma\left(n-2c\right)^{2}}{\Gamma\left(n-2c+1\right)\Gamma\left(n-2c-1\right)} \tag{11}$$

iii. Precautionary Loss Function Using extended Jeffrey Prior.

The precautionary loss function associated with the parameter α is defined as;

$$L\left(\alpha_{PLF},\alpha\right)=\frac{\left(\alpha-\alpha_{PLF}\right)^{2}}{\alpha} \tag{12}$$

Where α_{PLF} is the estimator of the parameter using PLF and it is obtained as;

$$\alpha_{PLF}=\sqrt{\frac{\Gamma\left(n-2c+3\right)}{\Gamma\left(n-2c+1\right)\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{2}}} \tag{13}$$

And the risk is

$$p\left(\alpha_{PLF}\right)=2\left(\frac{\sqrt{\Gamma\left(n-2c+3\right)}\Gamma\left(n-2c+1\right)-\Gamma\left(n-2c+2\right)}{\Gamma\left(n-2c+1\right)^{2}\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)}\right) \tag{14}$$

3.3 Bayes Estimation Using Quasi Prior Under Various Loss Function And Its Corresponding Risk

Posterior distribution assuming Quasi prior

The Quasi prior is given as $p\left(\alpha\right)\propto\frac{1}{\alpha^{c}}, \quad \alpha>0$

and the posterior distribution of parameter α is obtained as

$$p\left(\alpha/\underline{x}\right)=\frac{\alpha^{n-c}e^{-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}}\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{n-c+1}}{\Gamma\left(n-c+1\right)} \tag{15}$$

i. Squared Error Loss Function (SELF) using quasi prior.

The squared error loss function associated with the parameter α is defined as;

$$L\left(\alpha,\alpha_{SELF}\right)=\left(\alpha-\alpha_{SELF}\right)^{2} \tag{16}$$

Where α_{SELF} is the estimator of the parameter using SELF and it is obtained as;

$$\alpha_{SELF}=\frac{\Gamma\left(n-c+2\right)}{\Gamma\left(n-c+1\right)\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} \tag{17}$$

And the risk is

$$p\left(\alpha_{SELF}\right)=\frac{\Gamma\left(n-c+1\right)\Gamma\left(n-c+3\right)-\left(\Gamma\left(n-c+1\right)\right)^{2}}{\left(\Gamma\left(n-c+1\right)\right)^{2}-\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{2}} \tag{18}$$

ii. Quadratic Loss Function Using Quasi Prior.

The quadratic loss function (QLF) associated with the parameter α is defined as;

$$L\left(\alpha,\alpha_{QLF}\right)=\left(\frac{\alpha-\alpha_{QLF}}{\alpha}\right)^{2} \tag{19}$$

Where α_{QLF} is the estimator of the parameter using QLF and it is obtained as;

$$\alpha_{QLF}=\frac{\Gamma\left(n-c\right)}{\Gamma\left(n-c-1\right)\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}} \tag{20}$$

and the risk is

$$p\left(\alpha_{QLF}\right)=\frac{\Gamma\left(n-c+1\right)\Gamma\left(n-c-1\right)-\left(\Gamma\left(n-c\right)\right)^{2}}{\Gamma\left(n-c+1\right)\Gamma\left(n-c-1\right)} \tag{21}$$

iii. Precautionary Loss Function Using extended Jeffrey Prior.

The precautionary loss function associated with the parameter α is defined as;

$$L\left(\alpha_{PLF},\alpha\right)=\frac{\left(\alpha-\alpha_{PLF}\right)^{2}}{\alpha} \tag{22}$$

Where α_{PLF} is the estimator of the parameter using PLF and it is obtained as;

$$\alpha_{PLF}=\sqrt{\frac{\Gamma\left(n-c+3\right)}{\Gamma\left(n-c+1\right)\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)^{2}}} \tag{23}$$

and the risk is

$$p\left(\alpha_{PLF}\right)=2\left(\frac{\sqrt{\Gamma\left(n-c+3\right)}\Gamma\left(n-c+1\right)-\Gamma\left(n-c+2\right)}{\Gamma\left(n-c+1\right)^{2}\left(\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right)}\right) \tag{24}$$

4. **Results**

Table 1. The maximum likelihood and Bayesian analysis

Sample sizes	Measures	MLE	Extended Jeffrey's prior			Quasi prior		
			SELF	QLF	PLF	SELF	QLF	PLF
25	Estimates	.4437	.4437	.4082	.4525	.4526	.4171	.4613
	Biases	.0840	.0840	.0818	.0866	.0866	.0811	.0899
	MSE	.0123	.0123	.0105	.0133	.0133	.0106	.0145
50	Estimates	.4398	.4398	.4222	.4442	.4442	.4266	.4486
	Biase	.0580	.0580	.0575	.0589	.0589	.0572	.0600
	MSEs	.0056	.0056	.0052	.0059	.0059	.0052	.0061
100	Estimates	.5447	.5447	.5284	.5475	.5475	.5366	.5502
	Biases	.0403	.0403	.0402	.0406	.0406	.0401	.0410
	MSEs	.0027	.0027	.0026	.0027	.0027	.0026	.0027
150	Estimates	.4595	.4595	.4534	.4611	.4611	.4549	.4626
	Biase	.0327	.0327	.0326	.0329	.0329	.0325	.0331
	MSE	.0017	.0017	.0017	.0017	.0017	.0017	.0018

Table 1. Shows the results obtained from the simulation. The Bayes estimates, biases, mean square error (MSE) were obtained under different sample sizes across the different estimators (maximum likelihood, SELF, QLF, PLF under extended Jeffery and quasi prior). where $\alpha = 0.5$ assuming $\beta, \lambda\ and\ c$ are known, given $\beta = \lambda = 1$ and $c = 0.5$. It is observed that, at sample size 25, the QLF has the smallest MSE of 0.0105 and 0.0106 under extended Jeffery and Quasi prior among other MSE from other estimators. Also at sample size 50, the QLF has the smallest MSE of 0.0052 and 0.0052 under extended Jeffery and Quasi prior among other MSE from other estimators. Similarly, the same apply to sample size 100. However, at the sample size 150, the MSE of all the estimators converges and are almost the same. This indicates that at a very large sample size the MSE of all the estimators converge and are the same.

Table 2. The maximum likelihood and Bayesian analysis

Sampl e sizes	Measures	MLE	Extended Jeffrey's prior			Quasi prior		
			SELF	QLF	PLF	SELF	QLF	PLF
25	Estimates	1.3311	1.3311	1.2247	1.3575	1.3578	1.2513	1.3841
	Biases	.2521	.2521	.2454	.2598	.2600	.2433	.2698
	MSE	.1106	.1106	.0945	.1197	.1198	.0955	.1308
50	Estimates	1.3195	1.3195	1.2667	1.3325	1.3327	1.2799	1.3458
	Biases	.1740	.1740	.1724	.1765	.1766	.1715	.1800
	MSE	.0506	.0506	.0468	.0527	.0526	.0470	.0553
100	Estimates	1.6342	1.6342	1.6016	1.6424	1.6424	1.6097	1.6506
	Biases	.1210	.1210	.1206	.1218	.1218	.1202	.1229
	MSE	.0238	.0238	.0230	.0243	.0243	.0230	.0249
150	Estimates	1.3786	1.3786	1.3602	1.3832	1.3832	1.3648	1.3878
	Biases	.0981	.0981	.0977	.0986	.0986	.0976	.0992
	MSE	.0154	.0154	.0150	.0156	.0156	.0150	.0159

Table 2. Shows the results obtained from the simulation. The Bayes estimates, biases, mean square error (MSE) were obtained under different sample sizes across the different estimators (maximum likelihood, SELF, QLF, PLF under extended Jeffery and quasi prior). where $\alpha = 1.5$ assuming $\beta,\ \lambda\ and\ c$ are known, given $\beta = \lambda = 1$ and $c = 0.5$. It is observed that, at sample size 25, the QLF has the smallest MSE of 0.0945and 0.0955 under extended Jeffery and Quasi prior among other MSE from other estimators. Also at sample size 50, the QLF has the smallest MSE of 0.0468 and 0.0470under extended Jeffery and Quasi prior among other MSE from other estimators. Similarly, the same apply to sample size 100 and 150.

Table 3. The maximum likelihood and Bayesian analysis

Sample sizes	Measures	MLE	Extended Jeffrey's prior			Quasi prior		
			SELF	QLF	PLF	SELF	QLF	PLF
25	Estimates	2.2186	2.2185	2.0410	2.2625	2.2629	2.0855	2.3069
	Biases	.4202	.4202	.4090	.4330	.4332	.4056	.4497
	MSE	.3073	.3073	.2625	.3324	.3327	.2653	.3633
50	Estimates	2.1991	2.1551	2.0672	2.1770	2.1991	2.1111	2.2210
	Biases	.2899	.2859	.2947	.2872	.2900	.2874	.2942
	MSE	.1404	.1326	.1328	.1358	.1404	.1300	.1463
100	Estimates	2.7237	2.6965	2.6420	2.7101	2.7237	2.6693	2.7373
	Biases	.2016	.2003	.2038	.2007	.2016	.2010	.2030
	MSE	.0662	.0643	.0645	.0651	.0662	.0638	.0675
150	Estimates	2.2977	2.2824	2.2517	2.2900	2.2977	2.2671	2.3053
	Biases	.0427	.0419	.0419	.0422	.0427	.0416	.0434
	MSE	.0427	.0419	.0419	.0422	.0427	.0416	.0434

Table 3. Shows the results obtained from the simulation. The Bayes estimates, biases, mean square error (MSE) were obtained under different sample sizes across the different estimators (maximum likelihood, SELF, QLF, PLF under extended Jeffery and quasi prior). where $\alpha = 2.5$ assuming $\beta,\ \lambda\ and\ c$ are known, given $\beta = \lambda = 1$ and $c = 0.5$. It is observed that, at sample size 25, the QLF has the smallest MSE of 0.2625 and 0.2653 under extended Jeffery and Quasi prior among other MSE from other estimators. Also at sample size 50, the QLF has the smallest MSE of 0.1328 and 0.1300 under extended Jeffery and Quasi prior among other MSE from other estimators. Similarly, the same apply to sample size 100 and 150.

Conclusion

The table 1, 2 and 3 above show the estimates, biases and mean square errors of the classical method of estimating the parameter (shape) of the distribution (maximum likelihood) and the Bayesian analysis. The results show that quadratic loss functions under extended Jeffrey prior and quasi prior Outperformed the squared error loss function and the precautionary loss function across different sample sizes. The result also reveals that the Bayesian estimates of the parameter (shape) under extended Jeffrey and quasi prior using quadratic loss function is better than the maximum likelihood Estimate. However, the quadratic loss function under the quasi prior performed better than the quadratic loss function under the extended Jeffery prior.

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