ODD TRANSMUTED RAYLEIGH- LOMAX DISTRIBUTION WITH ITS APPLICATIONS

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ABSTRACT

The flexibility and applicability of some known compound distributions in modelling and predicting diverse industrial datasets cannot be quantified. Nowadays, some of these existing distributions found it difficult in modelling different datasets most especially the ones that are highly skewed to either direction. On this note, this study focused on developing a distribution named Odd Transmuted Rayleigh- Lomax Distribution (OTRLD) which can capture the important data points. The cumulative distribution function (CDF), probability density function (PDF) and reliability function of the OTRLD were explicitly derived. Consequently, the validity of this distribution has been checked and confirmed valid. The parameters of the new distribution were estimated using the technique of maximum likelihood. On the basis of distributional performance, the derived model was compared with existing distributions in the literature via real datasets. The results have shown that the proposed distributions outperformed their competitors.

Keywords: Odd Transmuted Rayleigh, Lomax, reliability, maximum likelihood.

1. Introduction

Statistical distributions are found to be useful in terms of future prediction modelling real datasets. Tahir *et al.,* (2015) indicated that theoretical and applied statisticians developed interest in proposing generalized families of distributions due to the flexibility of their properties. Recent studies in statistics have revealed that the methods for generating new probability distributions are commonly referred to as generators, families, or generalized classes of probability distributions (Mohammed & Ugwuowo, 2020).

Some notable researchers have extended Lomax distributions using different families of distributions and these distributions played a vital role in analysing skewed datasets see; Marshall-Olkin Alpha Power Lomax Distribution by Almongy et al., (2021), Extended form of Lomax by Alsuhabi, et al., (2022), Nadarajah–Haghighi Lomax distribution by Nagarjuna et al., (2022), Bivariate power Lomax distribution by Qura et al. (2023). Kajuru et al. (2023a) came up with generator that can be used to create different forms of distributions which can better fit real-world datasets with different shapes and another powerful family involving Gompertz distribution was extensively studied by (Kajuru et al. 2023b).

Although numerous distributions have been introduced and applied in modelling varying dataset, numerous methods for developing new distributions new distributions have been established in the literature which are found to be more flexible for fitting different industrial datasets (Ogunde et al. 2023). These techniques result in the creation of new distributions that extend the scope of already existing distributions and provide greater flexibility in modeling real-world datasets. In some cases, the use of expanded distributions has resulted in better parametric fits for certain datasets (Abdullahi et al., 2023).

2. Odd Transmuted Rayleigh- Lomax Distribution (OTRLD)

The Lomax distribution, also known as the Pareto Type II distribution, is a probability distribution used to model heavy-tailed data and is a variation of the of the Pareto distribution. It is characterised by two parameters namely shape (β) and scale (α). The distribution is defined for positives and has a PDF that decays more slowly than the exponential distribution. It's commonly used in various field, including finance, economics and engineering, to model extremes events and rare occurrences.

The CDF and PDF are respectively given as;

$$G(x,\alpha,\beta)=1-\left(1+\frac{x}{\beta}\right)^{-\alpha} \tag{1}$$

$$g(x,\alpha,\beta)=\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)} \tag{2}$$

Abdullahi et al. (2023) derived the cdf and pdf of OTR-X family of distributions and are given below;

$$F(x,\sigma,\lambda,\phi)=\left(1-e^{-\frac{1}{2\sigma^2}\left(\frac{G(x,\phi)}{\bar{G}(x,\phi)}\right)^2}\right)\left(1+\lambda e^{-\frac{1}{2\sigma^2}\left(\frac{G(x,\phi)}{\bar{G}(x,\phi)}\right)^2}\right) \tag{3}$$

$$f(x,\sigma,\lambda,\phi)=\frac{g(x;\phi)}{\sigma^2}\frac{G(x;\phi)}{\left(\bar{G}(x;\phi)\right)^3}e^{-\frac{1}{2\sigma^2}\left(\frac{G(x;\phi)}{\bar{G}(x;\phi)}\right)^2}\left[1-\lambda+2\lambda e^{-\frac{1}{2\sigma^2}\left(\frac{G(x;\phi)}{\bar{G}(x;\phi)}\right)^2}\right] \tag{4}$$

Now, the CDF and PDF of the novel distribution termed as Odd Transmuted Rayleigh Lomax Distribution (OTRLD) generated from the OTR-X family of distributions are presented in equation (5) and (6) respectively.

$$F\left(x,\sigma,\lambda,\alpha,\beta\right)=\left(1-e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right)\left(1+\lambda e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right) \quad (5)$$

$$f\left(x,\sigma,\lambda,\alpha,\beta\right)=\frac{\left(\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}\right)}{\sigma^2}\frac{\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)}{\left(\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^3}e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\left[1-\lambda+2\lambda e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right] \quad (6)$$

2.1 Limiting Behaviour

$$\lim_{x\to 0}F\left(x,\sigma,\lambda,\alpha,\beta\right)=\lim_{x\to 0}\left(1-e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right)\left(1+\lambda e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right)=0$$

And

$$\lim_{x\to \infty}F\left(x,\sigma,\lambda,\alpha,\beta\right)=\lim_{x\to \infty}\left(1-e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right)\left(1+\lambda e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right)=1$$

The results above prove that the distribution is valid.

Figures 1 and 2 show the behaviour of the CDF and PDF of the OTRLD for a few varying values of the parameters and furthermore, for a simplicity, $\sigma=a$, $\lambda=b$, $\alpha=d$ .and $\beta=e$.

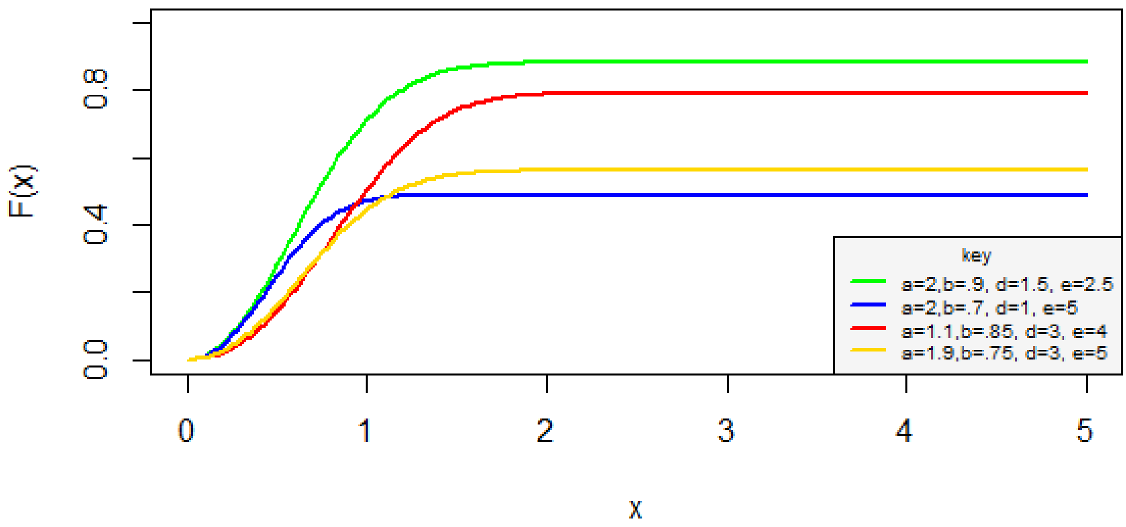

Figure 1: The CDF plot of OTRLD

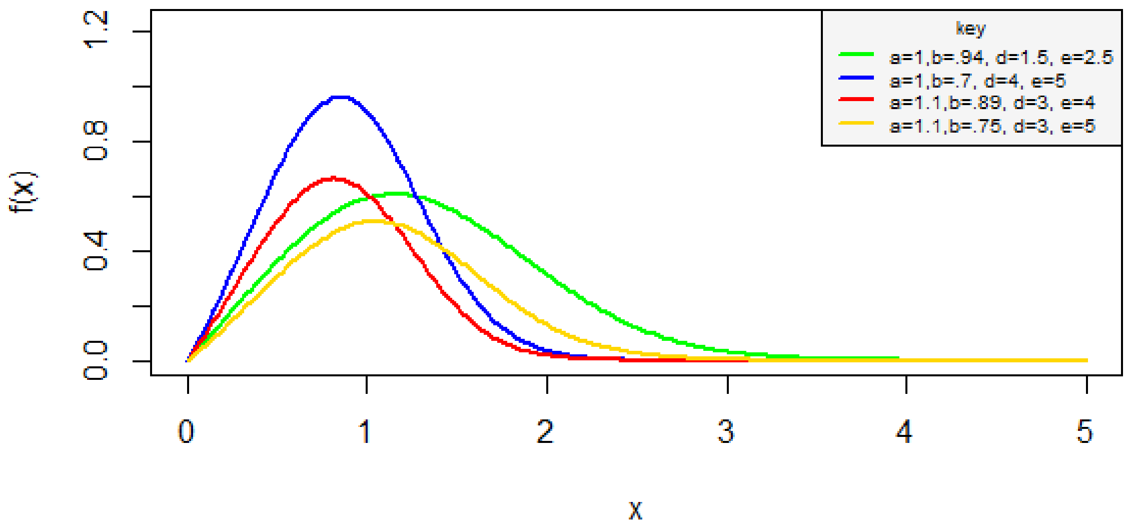

Figure 2: The PDF plot of OTRLD

The Survival function of OTRLD is derived as:

$$S(x,\sigma,\lambda,\alpha,\beta)=1-\left(\left(1-e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right)\left(1+\lambda e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right)\right) \tag{7}$$

The Hazard function of OTRLD is derived as:

$$h(x,\sigma,\lambda,\alpha,\beta)=\frac{\dfrac{\left(\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}\right)}{\sigma^2}\dfrac{\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)}{\left(\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)^3}e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\left[1-\lambda+2\lambda e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right]}{1-\left(\left(1-e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right)\left(1+\lambda e^{-\frac{1}{2\sigma^2}\left(\frac{1-\left(1+\frac{x}{\beta}\right)^{-\alpha}}{\left(1+\frac{x}{\beta}\right)^{-\alpha}}\right)^2}\right)\right)} \tag{8}$$

The behaviour of the survival and hazard functions of the OTRLD for a few varying parameters' values are respectively display below;

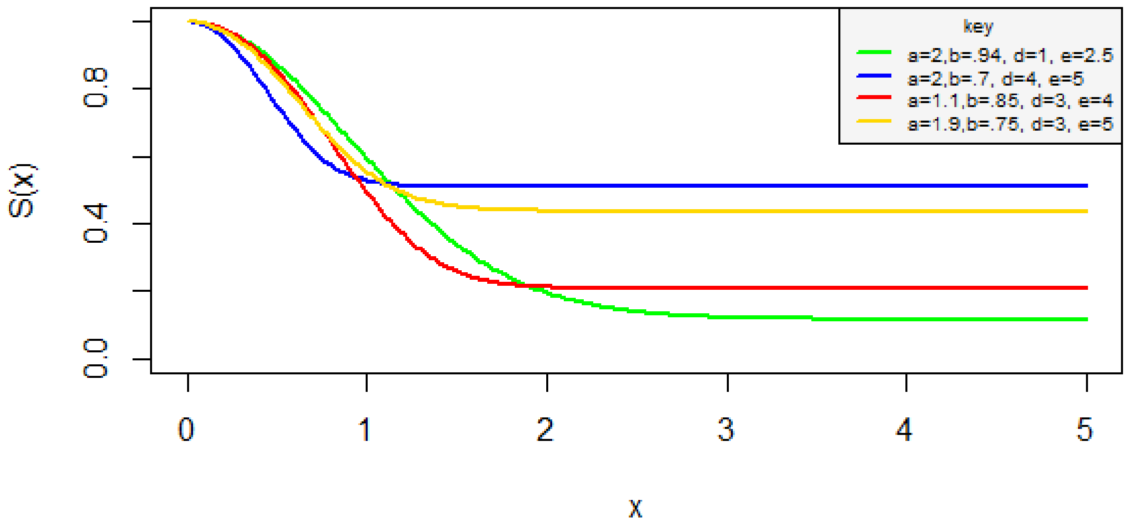

Figure 3: The Survival Function plot of OTRLD

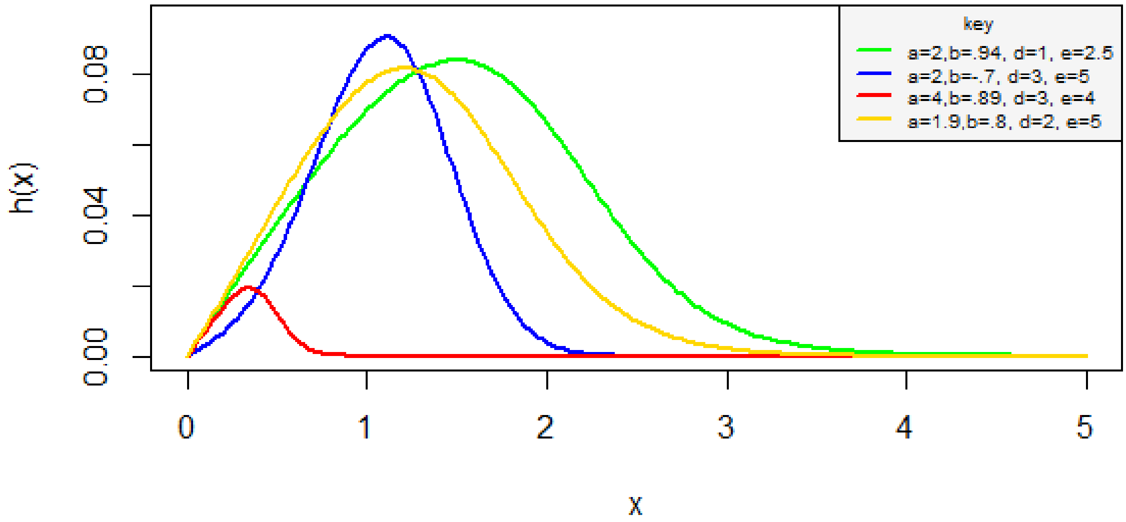

Figure 4: The Hazard Function plot of OTRLD

3. Estimation of Parameters of the OTRLD

Here, the new generated distribution's parameters are computed using the maximum likelihood estimation technique. With sample values $x_1, x_2, ..., x_n$ with joint PDF as $f(x_1, x_2, ..., x_n; \Theta)$, where $\Theta = (\sigma, \lambda, \eta, \alpha)^T$ is a vector of an unknown parameter, let $X_1, X_2, ..., X_n$ be n RV, from a population that follows OTRLD. The ℓ is therefore described as follows:

$$\ell=-2n\log\sigma+\sum_{i=1}^{n}\log\left(\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(\alpha+1)}\right)+\sum_{i=1}^{n}\log\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)-3\sum_{i=1}^{n}\log\left(\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)$$
$$-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\frac{\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)}{\left(\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)}\right)^2+\sum_{i=1}^{n}\log\left(1-\lambda+2\lambda e^{-\frac{1}{2\sigma^2}\left(\frac{\left(1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)}{\left(\left(1+\frac{x}{\beta}\right)^{-\alpha}\right)}\right)^2}\right) \quad (9)$$

By performing partially differentiation for ℓ with regard to λ,σ, β and α and equating the result to 0, one can derive the component of the score vector, $U(\Theta)$.

4. Applications

4.1 Datasets

The first dataset, which reflects the number of years a group of patients who got only chemotherapy survived, is a part of the data described by the scholars named (Abdullahi et al., 2023). This dataset contains the times of survival expressed in years for 46 patients. The details are as follows: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

The second dataset used is the dataset applied by Shabbir et al. (2018) and it consists of seventy-two observations as: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38,43, 44, 48, 52, 53, 54,54, 55, 56, 57, 58, 58, 59, 60, 60, 60,60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

4.2 The OTRLD and Competing Models

To illustrate the performance of the OTRLD it was fitted to two different real-life datasets. The performance of the generated OTRG was compared to some existing distribution in the literature. The distributions are: The Transmuted Lomax Distribution (TLD), Rayleigh Lomax Distribution (RLD), and Lomax Distribution (LD).

Table 1: Estimates of the OTRLD and the competing models' parameters Data I

Model	σ	λ	$\hat{\alpha}$	$\hat{\beta}$
OTRLD	0.0044	0.4674	1.9039	0.2638
TLD	-	0.3165	0.9625	0.7628
RLD	-	-	0.8831	0.2958
LD	-	-	1.6958	0.4065

Table 2: Results of the Goodness-of-Fit Statistics for Data I

Model	$-\ell$	***AIC***	***CAIC***	***HQIC***	***BIC***	**RANK**
OTRLD	-772.8514	-1537.7030	-1536.7030	-1535.0090	-1530.4760	1
TLD	-720.1203	-1434.2406	1433.6552	-1438.9306	-1435.2810	2
RLD	79.3785	164.7570	165.3424	160.0669	163.7166	3
LD	83.7599	171.5197	171.8054	172.8667	175.1330	4

Table 3: Estimates of the OTRGD and competing models' parameters for Data II

Model	σ	λ	α	$\hat{\beta}$
OTRLD	0.6724	0.9359	0.5601	0.9031
TLD	-	0.8740	0.3814	1.0013
RLD	-	-	0.6705	1.9741

LD			0.4886	0.3841

Table 4: Results of the Goodness-of-Fit Statistics for Data II

Model	-LL	***AIC***	***CAIC***	***HQIC***	***BIC***	**RANK**
OTRLD	-430.5631	-855.1262	854.7733	-859.5128	-855.5542	1
TLD	-324.8612	-636.4290	-636.0761	-640.8157	-636.8570	2
RLD	95.9631	197.9262	198.2791	193.5395	197.4982	3
LD	128.7623	261.5246	261.6985	258.6001	261.2393	4

Tables 1 and 3 show the parameter estimates of the derived OTRLD with its competing distributions (TLD, RLD and LD) using two real datasets (data I and II). While tables 2 and 4 revealed that the OTRLD has the minimum values of AIC, CAIC and HQIC compared with the existing distributions considered, this implies that based on the information selection criteria the new distribution performed better in terms of model fitting for both data I and II.

5. Conclusion

This generalization of existing probability distributions has received reasonable attention due to the fact that, the generalized distributions are more flexible and performed well in terms of best fit for the datasets than their competitors with a smaller number of parameters. In comparison to existing distributions the OTRLD proven to be the best model that fitted the different datasets in the study.

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