MODELING CHAOTIC BEHAVIOUR OF STOCK PRICE INDEX USING ITO STOCHASTIC DIFFERENTIAL EQUATION

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Abstract

Stochastic differential equation (SDE) have become an important tool for modeling the dynamics of many random phenomena such as financial assets. In real applications, parameters of the equation are unknown and need to be estimated and many times only discretely sampled data of the process are available. Financial assets such as stock price are very chaotic and dynamic and are often represented using stock price index, to reflect overall market sentiments and directions of stock prices. Investing in stocks or equities is a speculative risk that is complex and complicated to understand due to its chaotic behaviour. In this paper, attempt was made to study this chaotic bahaviour via Ito SDE, the forward Kolmogorov equation (FKE). The parameters estimation was done using Euler-Maruyama method. The model's mean, variance and Akaike Information Criterion (AIC) were obtained as 0.08, 896.56 and 4764.08 respectively, as against ARIMA (1,0,0), (3,1,1) and (6,0,0) having AIC values of 5482.92, 5401.00 and 5433.50, respectively. Hence the Ito SDE was better in describing stock price index and is therefore recommended for practitioners and policy makers for sound decision making regarding stocks.

***Keywords**:* Chaos; Diffusion process; Kolmogorov equation; nonlinear dynamics; stochastic differential equation; Stock price index.

1. Introduction

Chaos, the science of nonlinear systems has provided a new set of tools for understanding the prediction of random behaviour in time series modelling. The body of chaotic models is fascinating since time series data that seems random may in reality be chaotic. Chaos theory as a branch of mathematics focuses on the behaviour of dynamical systems that are highly sensitive to initial conditions.

Most economic and financial systems (or processes) are chaotic time series since their evolution appears disorderly and the linear stochastic approach of modelling and forecasting is not adequate for such random processes (Boaretto et al, 2021). Chaotic behaviour exists in many natural systems, such as weather and climate and also occurs spontaneously in some systems with artificial components such as road traffic. This behaviour can be studied through analysis of a chaotic mathematical model, or through analytical techniques such as recurrence plots and Poincare maps and these has provided a new set of tools for understanding the prediction of random behaviour in time series modelling (Sandubete and Escot, 2020). Modeling and predicting the future evolution of a given time series from a chaotic dynamical system is one of the main tasks of nonlinear time series analysis (Agwuegbo et al, 2011).

A stochastic differential equation (SDE) is a mathematical equation relating a stochastic process to its local deterministic and random components. According to Oksendal (2003) and Karatzas and Shreve (1991), stochastic differential equation is defined as a white noise driven differential equation in which one or more of the terms is a stochastic process. The theory of stochastic differential equations (SDEs) has been extensively developed and is discussed in many books including Papoulis (1984), Karatzas and Shreve (1994), Friedman (2006) and Davis and Mikosch (2008). Stochastic differential equations are used to model diverse phenomena such as stock prices or physical systems subject to thermal fluctuations. Typically, SDEs incorporate white noise which can be thought of as the derivative of Brownian motion (or the Weiner process). Brownian motion is a Gaussian process and is considered as a very good approximation to many real-life phenomena (Mikosch, 1998).

Brownian motion is a fundamental building block of modern quantitative finance and indeed the basic model for financial asset prices. The essence of stochastic differential equations in this study is to reconstruct possible chaotic behaviour of financial time series examined in financial markets. The explicit solutions of the stochastic differential equations are in terms of the coefficient of the underlying Wiener and the diffusion processes. Diffusion processes is also a solution to stochastic differential equations (SDEs) and are primarily used as approximations to discrete processes (Zitkovic, 2016). These approximations sometimes can be solved explicitly when the motivating discrete model is intractable (Heyman and Sobel, 2004). Loosely speaking, the term diffusion (Karatzas and Shreve, 1994) is attributed to a Markov process which has continuous sample paths and can be characterized in terms of its infinitesimal generator.

Stochastic differential equations (SDEs) have become an indispensable ingredient for modeling the dynamics of a variety of random phenomena such as the chaotic behavior of financial assets. In real applications the parameters of the equation are unknown and need to be estimated. In most cases what is available is only discretely sampled data of the process and then it is a common practice to use the discretization of the original continuous time process for the modeling. Unlike deterministic models such as ordinary differential equations, which have a unique solution for each appropriate initial condition, SDEs have solutions that are continuous-time stochastic processes. This is a major motivation for SDEs in modeling chaotic time series, in which interest is in nonlinear dependence of the level of a series on previous data points (Allen, 2007).

In financial markets, stochastic processes occur whenever dynamical systems experience random influences. Mathematical models are useful for understanding these chaotic behaviours in financial processes. The chaotic behaviour and characteristics are often observed and commonly encountered in different fields of economics and finance, particularly in the capital markets. Financial market analysts are always looking for explanations of large movements in asset prices and one explanation is that there is some (unanticipated) news which caused prices in commodities to drastically mark down the value of equities. Another explanation was that the financial market is a chaotic process which is characterized by occasional large movements (Hsieh, 1990).

There has been a growing interest in the use of diffusion for describing chaotic system such as stock price index since the variances changes through time. In particular, security prices in itself is a random process because of the actions of many different factors, both human and materials, which give rise to uncertainties in the system. The motivation of this study stems from Boaretto et al, (2021) and focused on the use of a diffusion process in the analysis of chaotic behaviour in Nigeria stock price index. Diffusion processes are important in several areas of science for modeling real life phenomena and can be characterized in terms of its infinitesimal generator (Karatzas, and Shreve, 1994).

Nonlinear dynamics is the branch of physics that studies systems governed by equations more complex than the linear forms. Nonlinear systems such as stock prices, inflationary rates, exchange rates, interest rates, appear chaotic, unpredictable or counter intuitive and yet their behavior is not random. Nonlinear dynamical systems describing changes in financial assets over time may appear chaotic and are difficult to solve (Bruce et al, 2017). The systems can commonly be approximated by linear equations (linearization) using a differential equation. In particular, a differential equation is linear if it is linear in terms of the unknown function and its derivatives. Complex behaviors that arise from deterministic nonlinear dynamic systems exhibit two special properties such as; sensitive dependence on initial conditions, and characteristic structure. In nonlinear dynamics, speaking about a dynamical system usually means to speak about an abstract mathematical system which is a model for such an entity.

2. Methodology

2.1 Chaotic Time Series Stochastic Model

Chaotic processes are random process that can be described mathematically as a set of dynamical differential equations. Suppose a scalar time series $\{X(t), t = 1,2,......,N\}$ is a measurement on a chaotic dynamical system in the state space. The scalar time series $\{X(t)\}$ is a stochastic process. A stochastic process $X = \{X(t)\}$ is a collection of random variables. More precisely, for every $t \in [0,\infty), X(t): \Omega \to R$ is a random variable corresponding to some infinite set of outcomes Ω. For every possible outcome $\omega \in \Omega,$ the stochastic process give rise to a trajectory $X_{\omega} : [0,\infty) \to R.$ The stochastic process for the chaotic series is considered to be a Markov process, and can be written as a function $X : [0,\infty) \times \Omega \to R,$ where the randomness of X is determined by the choice of $\omega \in \Omega.$. By letting X be any stochastic process and dt an infinitesimal time step, the increment of X over the time interval $[t, t+dt]$ is defined as

$$dX(t) = X(t+dt) - X(t). \qquad (1)$$

For any fixed t and dt, the increment $dX(t)$ is a random variable. If the increment $dX(s)$ and $dX(t)$ over disjoint intervals $[s+ds]$ and $[t+dt]$ are independent, then X has independent increments. If the distribution of every increment is normal, then X is a Gaussian process. Equipped with these definitions, the chaotic time series can be modelled by means of the general equation

$$aX(t) + \frac{dX(t)}{dt} = \mathrm{Z(t)} \qquad (2)$$

where a is a constant, and $Z(t)$ denotes continuous white noise. The behaviour of $Z(t)$ follows a wiener process. Wiener process is a particular type of Markov stochastic process describing the behaviour of the well-known Brownian motion. The Wiener process is a continuous-time stochastic process and is Gaussian. Brownian motion is the most common example of a Wiener process. More importantly, the mathematical models used to describe Brownian motion are the fundamental tools on which all financial asset pricing and derivatives pricing are based. If $\{X(t), t \geq 0\}$ is a Brownian motion, then the process $\{Z(t), t \geq 0\}$ defined by

$$Z(t) = \int_0^t X(s)ds \qquad (3)$$

is an integrated Brownian motion. The chaotic time series can be modelled as an integrated Brownian motion by assuming that the rate of change of $\{Z(t), t \geq 0\}$ follows a Brownian motion. Hence the rate of change is then

$$\frac{d}{dt}Z(t) = X(t). \qquad (4)$$

Hence

$$Z(t) = Z(0) + \int_0^t X(s)ds \qquad (5)$$

It follows from the fact that Brownian motion is a Gaussian process. As $\{Z(t), t \geq 0\}$ is a Gaussian, it follows that its distribution is characterized by its mean value and covariance function. When $\{X(t), t \geq 0\}$ is a standard Brownian motion, then

$$\mathrm{E}[Z(t)] = \mathrm{E}[\int_0^t X(s)ds]$$

$$\int_0^t \mathrm{E}[X(s)]ds = 0 \qquad (6)$$

for $s \leq t$

$$Cov[Z(s), Z(t)] = \mathrm{E}[Z(s), Z(t)]$$

$$= \mathrm{E}[\int_0^t X(u)du \int_0^s X(v)dv]$$

$$= \mathrm{E}[\int_0^t \int_0^s X(u)X(v)dudv$$

$$= \int_0^t \int_0^s \mathrm{E}[X(u)X(v)dudv$$

$$= \int_0^t \int_0^s \min(u,v)dudv$$

$$= \int_0^t udu + \int_0^s vdv$$

$$= S^2(\frac{t}{2} - \frac{s}{6}) . \quad (7)$$

2.2 Chapman Kolmogorov Formula

In the theory of Markovian stochastic processes in probability theory, the Chapman–Kolmogorov equation is an identity relating the joint probability distributions of different sets of coordinates on a stochastic process.

When the stochastic process under consideration is Markovian, the Chapman–Kolmogorov equation is equivalent to an identity on transition densities. In the Markov chain setting, one assumes that $i_1 < ... < i_n$. Then, because of the Markov property,

$$p_{i1}, \dots, \quad {}_{i_n}(f_1, \dots, f_n) = p_{i1}(f_1)p_{i1,i_1}(f_2|f_1) \cdots p_{i_n,i_{n-1}}(f_n|f_{n-1}), \quad (8)$$

where the conditional probability $p_{i;j}\left(f_i \middle| f_j\right)$ is the transition probability between the times $i > j$. So, the Chapman–Kolmogorov equation takes the form

$$p_{i_3;i_1}(f_3|f_1) = \int_{-\infty}^{\infty} p_{i_3;i_2}(f_3|f_2)p_{i_3;i_2}(f_2|f_1)df_2 \quad (9)$$

Informally, this says that the probability of going from state 1 to state 3 can be found from the probabilities of going from 1 to an intermediate state 2 and then from 2 to 3, by adding up over all the possible intermediate states 2.

When the probability distribution on the state space of a Markov chain is discrete and the Markov chain is homogeneous, the Chapman–Kolmogorov equations can be expressed in terms of (possibly infinite-dimensional) matrix multiplication, thus:

$$P(t+s) = P(t)P(s) \quad (10)$$

where *P*(*t*) is the transition matrix of jump *t*, i.e., *P*(*t*) is the matrix such that entry *(i,j)* contains the probability of the chain moving from state *i* to state *j* in *t* steps.

As a corollary, it follows that to calculate the transition matrix of jump *t*, it is sufficient to raise the transition matrix of jump one to the power of *t*, that is $P(t) = P^t$, which is the differential form of the Chapman–Kolmogorov equation and is known as master equation.

2.2.1 Backward and Forward Kolmogorov Equations

The Kolmogorov backward equation and its adjoint sometimes known as the Kolmogorov forward equation are partial differential equations that arise in the theory of continuous-time continuous-state Markov processes. Informally, the Kolmogorov forward equation addresses such problem as when there is information about the state *x* of a system at time *t* (namely a probability distribution $p_t(x)$); we want to know the probability distribution of the state at a later time $s > t$. The adjective 'forward' refers to the fact that $p_t(x)$ serves as the initial condition and the partial differential equation is integrated forward in time (in the common case where the initial state is known exactly, $p_t(x)$ is a Dirac delta function centered on the known initial state).

The Kolmogorov backward equation on the other hand is useful when we are interested at time *t* in whether at a future time *s* the system will be in a given subset of states *B*, sometimes called the *target set*. The target is described by a given function $u_s(x)$ which is equal to 1 if state *x* is in the target set at time *s*, and zero otherwise. In other words, $u_s(x) = 1_B$, the indicator function for the set *B*. We want to know for every state *x* at time t, $(t < s)$ what is the probability of ending up in the target set at time *s* (sometimes called the hit probability). In this case $u_s(x)$ serves as the final condition of the PDE, which is integrated backward in time, from *s* to *t*.

The backward Kolmogorov equation assumes that the system X_t evolves according to the stochastic differential equation

$$dX(t) = \mu(t,X)dt + \sigma(t,X)dW(t) \quad (11)$$

Then the backward Kolmogorov equation is as follows (according to Risken (1996))

$$\frac{\partial p(t,x)}{\partial t} = -\frac{\partial(\mu(t,x)p(t,x))}{\partial x} + \frac{1}{2}\frac{\partial^2\big(\sigma(t,x)p(t,x)\big)}{\partial x^2} \quad (12)$$

for $t \leq s$, subject to the final condition $p(t,s) = u_s(x)$, and the corresponding forward Kolmogorov equation is

$$\frac{\partial p(s,x)}{\partial s} = -\frac{\partial(\mu(s,x)p(s,x))}{\partial x} + \frac{1}{2}\frac{\partial^2(\sigma^2(s,x)p(s,x))}{\partial x^2} \quad (13)$$

for $s \geq t$, with initial condition $p(t,x) = p_t(x)$.

One property of homogeneous Markov process is the Chapman Kolmogorov formula relation:

$$p_{i,j}^{(l+n)} = \sum_{m=0}^{\infty} p_{i,m}^{(l)} p_{m,j}^{(n)} \ for\ l,n \geq 0 \qquad (14)$$

where $p_{i,j} = P\{X_{n+1} = j | X_n = i\}, i \geq 0, j \geq 0$ define the transition probabilities for discrete times $t_n = n\Delta t$ so that $t_{n+k} = t_n + t_k$.

But the property for a nonhomogeneous discrete stochastic process is contained in the forward Kolmogorov equation which is of interest in developing models with stochastic differential equation. Let $p_k(t) = P(X(t) = x_k)$ be the probability distribution at time t. Then, $p_k(t + \Delta t)$ satisfies

$$p_k(t + \Delta t) = p_k(t) +$$

$$\left[p_{k+1}(t)s(t,x_{k+1}) - p_k(t)\big(r(t,x_k) + s(t,x_k)\big) + p_{k-1}(t)r(t,x_{k-1})\right]\Delta t/\delta^2 \qquad (15)$$

As $\Delta t \rightarrow 0$, the discrete stochastic process approaches a continuous-time process. Then as $\Delta t \rightarrow 0$, $p_k(t)$ satisfies the initial-value problem:

$$\frac{dp_k(t)}{dt} = -\left(\frac{p_{k+1}(t)a(t,x_{k+1}) - p_{k-1}(t)a(t,x_{k-1})}{2\delta}\right) + \left(\frac{p_{k+1}(t)b(t,x_{k+1}) - 2p_k(t)b(t,x_k) + p_{k-1}(t)b(t,x_{k-1})}{2\delta^2}\right) (16)$$

for $k = \ldots, -2, -1, 0, 1, 2, \ldots$ where $\{p_k(0)\}_{k=-m}^{m}$ are known. Equations (16) are the forward Kolmogorov equation for the continuous-time stochastic process which approximates the partial differential equation

$$\frac{\partial p(t,x)}{\partial t} = -\frac{\partial(a(t,x)p(t,x))}{\partial x} + \frac{1}{2}\frac{\partial^2(b(t,x)p(t,x))}{\partial x^2} \qquad (17)$$

and corresponds to a diffusion Process having the stochastic differential equation (Allen, 2007)

$$dX(t) = a(t,X)dt + \sqrt{b(t,X)}dW(t) \qquad (18)$$

There exists a close relationship between the discrete stochastic process defined by the forward Kolmogorov equation and the continuous process defined by (18) in particular, for small Δt and δ, the probability distribution of the solution (18) will be approximately the same as the probability distribution of solutions to the discrete stochastic process.

We may then be able to construct a realistic discrete stochastic process model for the dynamical system under investigation. Then, an appropriate stochastic differential equation model is inferred from the above argument. The approach here is to develop a stochastic differential equation model by first constructing a discrete stochastic process model. As time is made continuous, the probability distribution of the discrete stochastic model approaches that of the continuous stochastic model.

2.2.2 Fokker-Planck Equations and Stochastic Dynamics

Fokker-Planck equations describe probability densities of state vectors of systems governed by stochastic differential equations of Ito or Stratonovich type. Here we illustrate how a dynamic equation described by an Ito stochastic differential equation generates a Fokker-Planck equation.

$$dx_i(t) = K_i[x(t)]dt + \sum_j g_{i,j}\,[x(t)]dw_j(t),$$

for $i = 1, \dots, n$. (19)

We let $u(\cdot)$ be an arbitrary function of an economic state vector x. We write its differenatial retaining terms of $o_p(dt)$ as:

$$d(u) = \sum_i \left(\frac{\partial u}{\partial x_i}\right) dx_i + \left(\frac{1}{2}\right) \sum_{i,j} (\partial^2 / \partial x_i x_j) dx_i dx_j$$

by substituting the expression for dx's in the above.

Using the relation $\langle dw_j \rangle = 0$ and $\langle dw_i dw_j \rangle = \epsilon dt \delta_{i,j}$, where the terms in the angle brackets denote the average over x, we note that, given

$$\langle u(x) \rangle = \int u(x)\, p(x, t | x_0, t_0) dx,$$

its time derivative

$$\frac{d\langle u \rangle}{dt} = \int u(x)\, \partial p(x, t | x_0, t_0) \partial t dt$$

is rewritten as

$$\frac{d\langle u(x) \rangle}{dt} = \sum_i \int u(x) \frac{\partial}{\partial x_i} [K_i(x) p] dx + \epsilon/2 \sum_{i,j} \int u(x) \frac{\partial^2}{\partial x_i \partial x_j} (\sum_m g_{i,m} g_{j,m} p)\, dx.$$

Since u is arbitrary, this relation yields the Fokker-Planck equation:

$$\frac{\partial p(t, x)}{dt} = -\sum_i \frac{\partial}{\partial x_i} [K_i(x) p] + (\epsilon/2) \sum \frac{\partial^2}{\partial x_i \partial x_j} \left(Q_{i,j} p\right) \qquad (20)$$

where we define the matrix Q by

$$Q_{i,j} = \sum_m g_{i,m} g_{j,m} \,.$$

With large initial population size, the forward Kolmogorov equation (20) satisfies the Fokker-Planck equation. The probability distribution $p(t,x)$ is the probability distribution to the Ito SDE

$$dX(t) = (b-d)X(t)dt + \sqrt{(b+d)X(t)}dW(t) \quad (21)$$

This implies that solutions to the stochastic differential equation (21) have approximately the same probability distribution as the discrete stochastic forward Kolmogorov equation and hence is a reasonable model for the dynamical process.

2.3 Parameter Estimation of Model Coefficients

As the exact solution to a stochastic differential equation is generally difficult to obtain, it is useful to be able to approximate the solution. Numerical methods like Euler's method can be applied to Itô stochastic differential equation in differential form

$$dX(t) = f\big(t, X(t)\big)dt + g\big(t, X(t)\big)dW(t) \quad (22)$$

for $0 \le t \le T$ with f and g called the drift and diffusion coefficients respectively.

For the Itô stochastic differential equation (22), a continuous piecewise linear approximation to the solution $X(t)$ from Euler's approximation is $\hat{X}(t)$ given as

$$\hat{X}(t) = \frac{X_i(t_{i+1}-t)}{\Delta t} + \frac{X_{i+1}(t-t_i)}{\Delta t} \quad (23)$$

for $t_i \le t \le t_{i+1}$ and $i = 0,1,2,\dots,N-1$ where $\{X_i\}_{i=0}^N$ is the Euler approximation to equation (22) at the $N+1$ nodal points $\{t_i\}_{i=0}^N$. This approximate solution is commonly plotted as trajectories of sample paths. The function $\hat{X}(t)$ is a continuous linear approximation to the solution $X(t)$.

2.4 Geometric Brownian Motion

Geometric Brownian motion is a stochastic process constructed and often used to model financial processes subject to random noise. Suppose $W = \{W_t : t \in [0,\infty)\}$ is standard Brownian motion and that $\mu \in \mathbb{R}$ and $\sigma \in (0,\infty)$. Let

$$X_t = \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right], \quad (24)$$

The stochastic process $X = \{X_t : t \in [0,\infty)\}$ is a geometric Brownian motion with drift parameter μ and volatility parameter σ.

We note that the stochastic process $\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t: t \in [0, \infty)\}$ is a Brownian motion with drift parameter $\mu - \frac{\sigma^2}{2}$ and scale parameter σ, so geometric Brownian motion is simply the exponential of this process. In particular, the process is always positive and this is one of the reasons it is used to model financial and other processes that cannot be negative.

Geometric Brownian motion $X = \{X_t: t \in [0, \infty)\}$ satisfies the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t \quad (25)$$

and for $t \in (0, \infty)$, X_t has the lognormal distribution with parameters $\left(\mu - \frac{\sigma^2}{2}\right)t$ and $\sigma\sqrt{t}$. This forms the basis for modeling the dynamics of stock prices.

This standard geometric Brownian motion is often assumed for stock price where the drift and diffusion coefficients are proportional to the initial stock price and it differs from our proposed approach already set forth. Specifically, stock price follows geometric Brownian motion if it satisfies the stochastic differential equation of the form equation (25).

Considering the model for financial variables, X_t over an infinitesimal period $[t, t + dt]$. According to Central limit theorem, the assumption is that X_t is normally distributed, with mean and variance both proportional to the time interval given by

$$\frac{dX_t}{X_t} \sim N(\mu dt, \sigma^2 dt), \qquad \mu, \sigma^2 \ constant$$

Now, since $dW_t \sim N(0, dt)$, it follows that $\mu dt + \sigma dW_t \sim N(\mu dt, \sigma^2 dt)$, one therefore take our model to be

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t \quad (26)$$

Then

$$dX_t = \mu X_t dt + \sigma X_t dW_t \quad (27)$$

This is a stochastic differential equation for the variable X_t. Given an initial value X_0, by first observing that the Ito's lemma implies that:

$$dlogX_t = \frac{dX_t}{X_t} - \frac{1}{2}\frac{(dX_t)^2}{X_t^2} = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t \quad (28)$$

Then

$$Log\ X_t = logX_0 + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t \quad (29)$$

Taking exponential of both sides, we have

$$X_t = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \tag{30}$$

The process is called the diffusion process with drift μ and volatility σ^2.

3 Results and Discussion

3.1 Data Description of the Chaotic System

Figure 1 below shows the time series plot of the Nigeria Stock price Index from 1993 to 2019 on a monthly basis. The series shows no seasonal component, although there is an increasing trend.

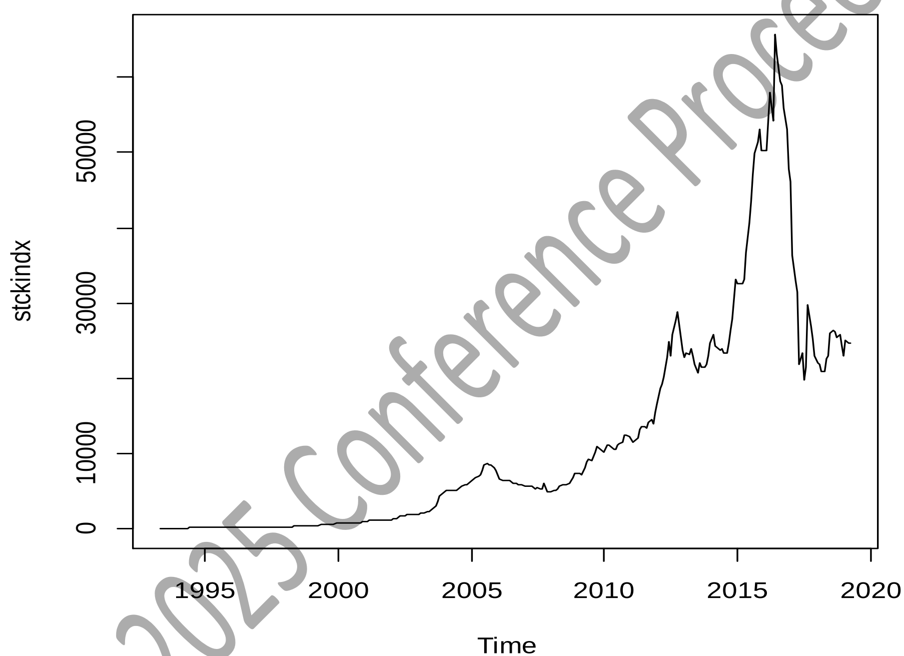

Figure 1. Time Series Plot of Stock Price Index

3.2 Selecting a Candidate ARIMA Model

An appropriate ARIMA model for the raw stock price index is arrived at as given in Table 1 below. The suitable model can be obtained with the "auto.arima" function in R. Table 4 shows the ARIMA model, ARIMA(3,1,1) of order 1 as the best choice for the stock price index with difference 1 and also the coefficients of the model and diagnostic statistics of the model.

Table 1: Coefficients of ARIMA of Order(3,1,1) for Stock Price Index Data

Coefficients	*ar1*	*ar2*	*ar3*	*ma1*
Value	-0.5846	0.2675	0.4090	0.7151
s.e	0.0731	0.0596	0.0517	0.0648

The AIC for the ARIMA is 5401 which gives the lowest AIC amongst the candidate models for the stock price index.

3.3 Predicting Stock Price Index with the ARIMA Model

Figure 2 shows the forecasts of stock price index, the blue line, using the ARIMA model of order (3,1,1). The 80% and 95% prediction intervals are shown by the grey and light grey shaded area around the blue line.

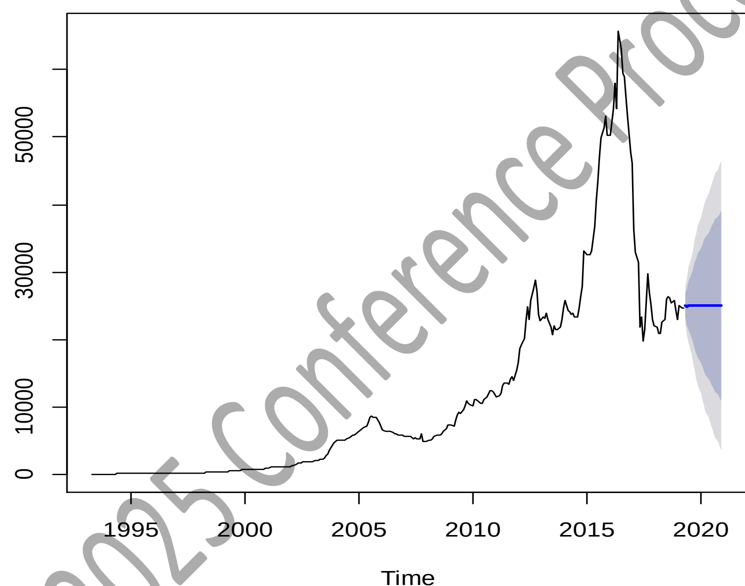

Figure 2. ARIMA (3,1,1) Forecast Plot of Stock Price Index

To investigate whether the forecast errors of the ARIMA model are normally distributed with mean zero and constant variance, and whether there are correlations between successive forecast errors, the correlogram of the forecast errors for our ARIMA(3,1,1) model for the stock price index is checked and the Ljung-Box test for lags 1-40 is performed.

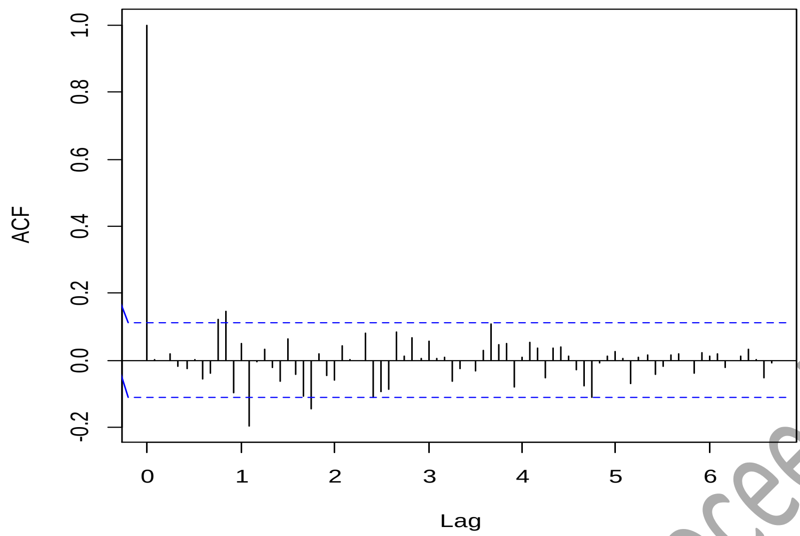

Figure 3. ACF Plot of Forecast Residuals of Stock Price Index

Since the correlogram shows that only lag 1 exceeds the significance bounds and the p-value for the Ljung-Box test is 0.142, it can be concluded that there is very little evidence for non-zero autocorrelations in the forecast errors at lags after 1.

The histogram and time plot (with overlaid normal curve) of the forecast errors show the forecast errors are normally distributed with mean zero and constant variance, The time plot of forecast errors shows that the forecast errors seem to have roughly constant variance over time and mean zero.

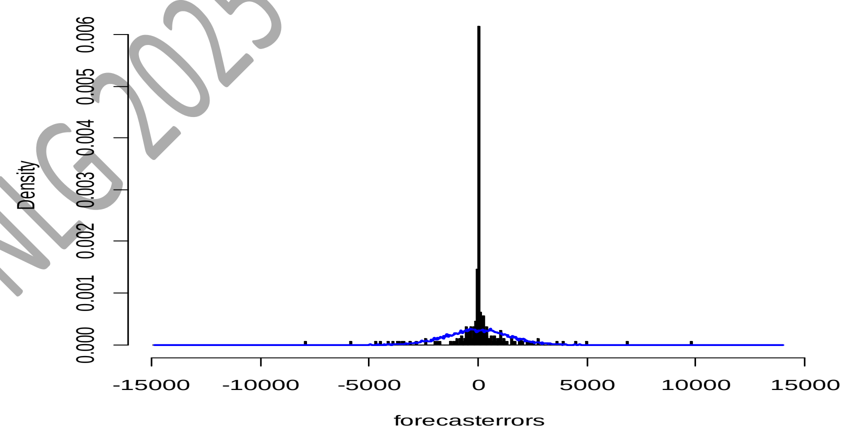

Figure 4. Histogram Plot of Forecast Residuals of Stock Price Index

The time plot of the in-sample forecast errors shows that the variance of the forecast errors seems to be roughly constant over time (though perhaps there is slightly higher variance for the second half of the time series). The histogram of the time series shows that the forecast errors are roughly normally distributed and the mean seems to be close to zero. Therefore, it is plausible that the forecast errors are normally distributed with mean zero and constant variance.

Since successive forecast errors do not seem to be correlated, and the forecast errors seem to be normally distributed with mean zero and constant variance, the ARIMA(3,1,1) does seem to provide an adequate predictive model for the stock price index.

3.4 Forward Kolmogorov Equation Model

The forward kolmogorov equation (FKE) defines the system under study and Table 2 below shows the coefficient of SDE model of the Ito type for the stock price index evaluated with Euler-Maruyama scheme implemented in R package, which is given as

$$dX_t = \theta_1 X_t dt + \sqrt{\theta_2 X_t} dW(t)$$

Table 2: Estimates of FKE Model for Stock Price Index

Coefficient	*Estimate*	*Std. Error*	*2.5%*	*97.5%*
θ_1	0.08233	0.05471	-0.02490	0.18956
θ_2	896.5623	71.911	755.62	1037.50

The FKE model is therefore

$$dX_t = 0.0823 * X_t dt + \sqrt{896.562 * X_t} dW(t) \tag{34}$$

with solution given as

$$\hat{X}_t = 111.3 * e^{(0.0823-448.281)t+29.94265W_t} \tag{35}$$

The solution in equation (35) is a function that is commonly plotted as simulated trajectories of sample paths. The 95% confidence interval of the estimates are given in the table and AIC is 4764.075 which is less than the AIC from the ARIMA models.

3.5 Diffusion Process of the Forward Kolmogorov Equation Model

Figure 5 shows the time plot of the simulated one trajectory of the diffusion process of the FKE in equation (35) for the stock price model. It can be seen that the FKE models the path of the stock price better with the increasing trend as observed in the actual times series plot of the data.

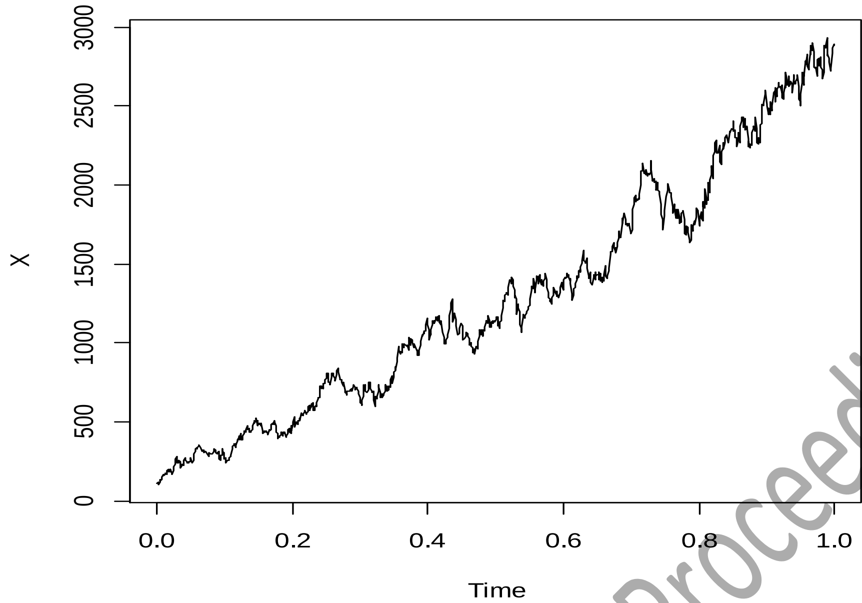

Figure 5. Simulated 1 trajectory of the Stock price Index process

$$dX_t = 0.0823 * X_t dt + \sqrt{896.562 * X_t} dW(t)$$

Figure 6 shows the plot of the simulation for 100 trajectories of the diffusion process with the 95% confidence interval shown by the blue lines and the mean path indicated by the red line.

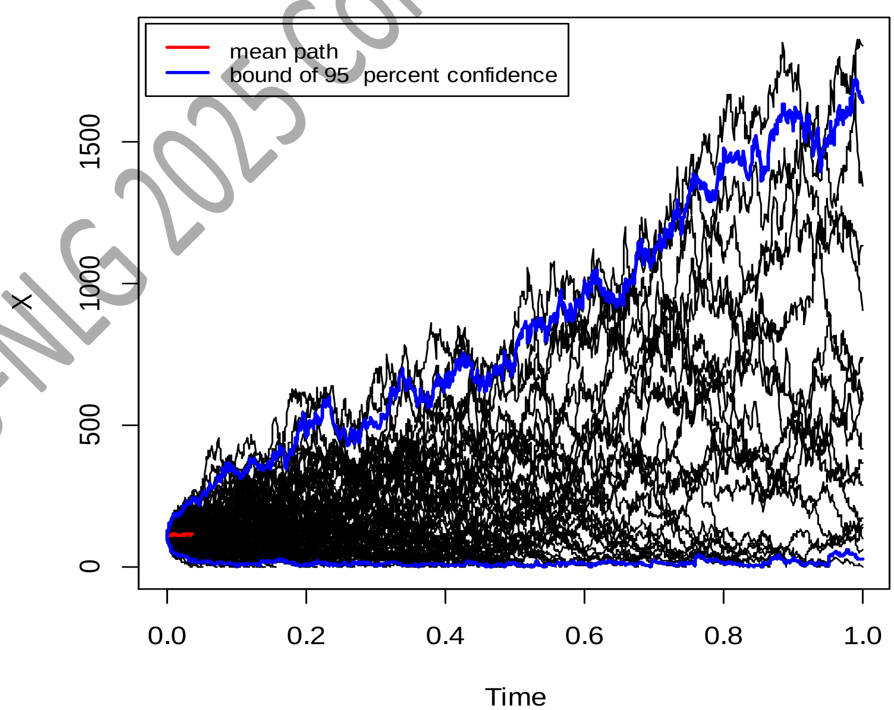

Figure 6. Simulated 100 trajectories of the Stock Price Index process

Table 3 shows the Monte-Carlo Statistics for diffusion process *X(t)* for the SDE model of stock price index obtained in equation (4.1).

Table 3. Monte-Carlo Statistics of the Diffusion Process X(t)

Statistics	*Value*
Mean	556.2
Variance	259213.6
Skewness	0.9949
Kurtosis	3.077
Coef-variation	0.91544

Figure 7 shows the histogram of the simulation and its distribution as also confirmed by the kernel density plot in Figure 8. Both plots show that the simulations are approximately normal and hence indicate a better fit of the FKE diffusion model for the stock price index with mean zero and constant variance 446.04.

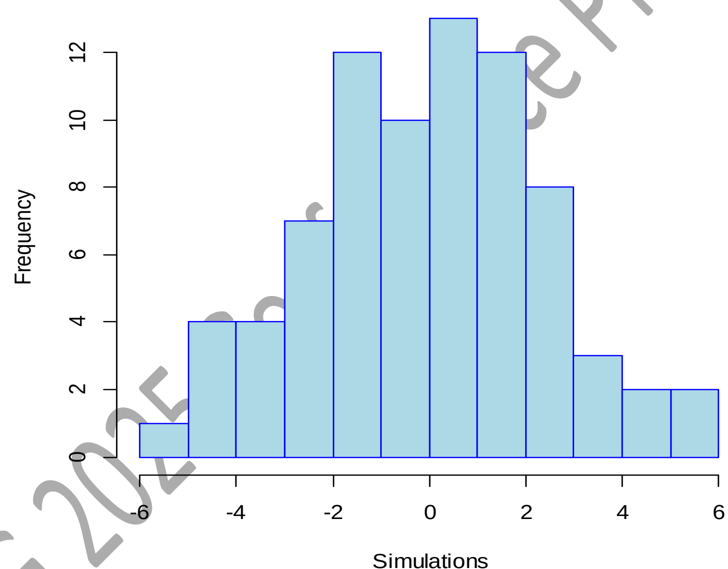

Figure 7. Histogram of the Simulation Values

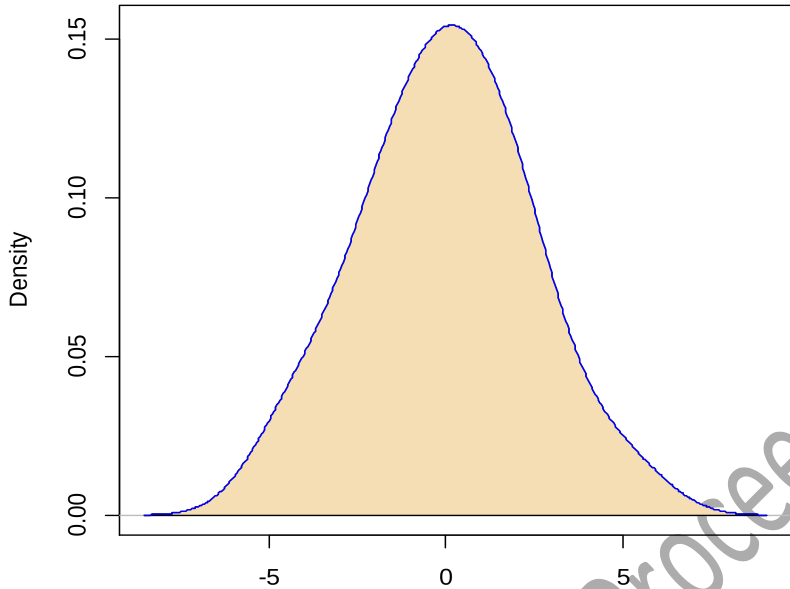

Figure 8. Estimated Density of Random Stock Price Index

3.6 Comparing the ARIMA and FKE Models

Models comparison with criteria such as Akaike Information Criterion (AIC) and Bayesian information criterion (BIC) are usually used. This study employs the AIC, where the minimum value rule of AIC is taken into consideration for selecting the best model to fit. Table 4 shows the FKE model has the minimum AIC.

Table 4: AICs of Fitted Models for Stock Price Index

Model	AR(6)	ARIMA(1,0,0)	ARIMA(3,1,1)	SDE
AIC	5433.50	5482.92	5401.00	4764.08

Hence, the stochastic differential equation diffusion process models the stock price index better than the conventional ARIMA models.

The following can therefore be drawn as conclusion from the data analysis of stock price index under stochastic differential equation modeling framework.

The stock price index modeled by ARIMA model of order (3,1,1) was best fitting. With stochastic differential equations model, using the forward Kolmogorov equation of the Ito type, the stock price index was better modelled than the ARIMA models as indicated by their AICs.

The diffusion process of the FKE model was solved numerically by multi-dimensional Euler-Maruyama scheme as implemented in R statistical package and the plot shown by the diffusion plot in Figure 6.

4. Conclusion

One of the basic economic problems facing every society whether developed or underdeveloped is the problem of stocks. Stock prices have enormous consequences on the economy and by extension on the investors. In particular, the stock prices in themselves are random process because of the actions of many different factors, both human and materials, which give rise to uncertainties in the system. Stock prices are amongst the most important economic indices affecting every economic system. It is related to indicators like exchange rates, interest rates, unemployment rate, gross domestic product and so on. Stocks influence all sectors of the economy all over the worlds since inflation is marked with increases in the prices of goods and services. It is always seen as the basic economic problem of every society. Several discrepancies have surfaced recently regarding the structure of stock prices and unemployment rates in Nigeria. These discrepancies can be translated into dubious national policies if they are left unchecked. These discrepancies in the structure of stock prices in Nigeria define controversial points as hypotheses in nonlinear dynamics.

Nonlinear dynamical systems describing changes in stock prices over time, may appear chaotic and are difficult to solve. The systems can commonly be approximated by linear equations (linearization) using a differential equation. Unlike deterministic models such as ordinary differential equations, which have a unique solution for each appropriate initial condition, SDEs have solutions that are continuous-time stochastic processes.

In summary, the procedure described was a stochastic differential model for a dynamical process with a discrete stochastic model which is the forward kolmogorov equation. The process carefully observes the possible changes along with the corresponding probabilities for a short time step Δt and as the short time approaches zero, the discrete model approaches the distribution of the continuous-time process corresponding to the forward kolmogorov equation. The procedure described provides, in a natural manner, an Ito stochastic differential equation model.

Nigeria stock price index from 1993 to 2019 was considered to change in a small time interval Δt. An ARIMA(3,1,1) model of order 1 was discovered as a better choice from the pool of ARIMA models. However, the FKE model defined by distribution of solutions to the Ito stochastic differential equation gave a better fit to the stock price index.

The Diffusion process for the FKE was simulated using the multi-dimensional Euler-Maruyama scheme for SDEs implemented in R statistical packages. The diffusion process of the FKE for stock price index on the actual data obtained showed that the diffusion process modeled the chaotic movement of the stock price index. The AIC of the FKE model was 4764.075, which was less than the AIC of the best ARIMA model of order(3,1,1) with AIC of 5401.

The study was able to advance a chaos analysis technique based on forward Kolmogorov equation for the continuous-time process corresponding to a Diffusion Process for the stochastic differential equation. The model built a stochastic differential model using continuous-time Markov process with available discrete time realisations of the chaotic process of the stock price index of Nigeria.

It is recommended that chaotic systems like stock price dynamics should be modeled using forward Kolmogorov equations that allow the approximation of a continuous time process using Markov chain through discrete realizations of the chaotic systems, as applied for stock prices. The modeling procedure can be extended to more than two stock prices for the purpose of financial portfolio analyses and management for decision making and competitive advantage. The modeling approach can be further extended to stochastic differential equations other than Ito type.

References

Agwuegbo, S. O. N., Solarin, A. R. T., and Asiribo, O. E. (2011). Hierarchical Structured Model for Nonlinear Dynamical Processes. *Journal of Nigerian Association of Mathematical Physics.* Vol 19, (November, 2011), pp15-20.

Allen, E. (2007). Modeling with Itˆo Stochastic Differential Equations. ISBN-13 978-1-4020-5952-0 (HB). Published by Springer, P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

Boaretto, B.R.R., Budzinski, R.C., Rossi, K.L., Prado, T.L., Lopes, S. R. and Masoller, C. (2021). Discriminating chaotic and stochastic time series using permutation entropy and artificial neural networks. Scientific Reports , (2021) 11:15789. https://doi.org/10.1038/s41598-021-95231-z

Bruce H., Nigel L. and Fernando C. (2017). Nonlinear Dynamics Time Series Analysis. Retrieved from http://onlinelibrary.wiley.com/doi/pdf on 14/06/2017

Davis, R. A. and Mikosch, T. (2008). The extremogram: A correlogram for extreme events. Technical Report. http://math.ku.dk/~mikosch.

Friedman A. (2006). Stochastic Differential Equations and Applications, Dover Publications Inc., Mineola, New York

Heyman, D.P. and Sobel, M.J. (2004). Stochastic Models in Operations Research, Vol. 1, Stochastic Processes and Operating Characteristics, Dover Publications inc Mineola, New York.

Hsieh, D. A. (1990). Chaos and Nonlinear Dynamics: Application to Financial Markets. Fuqua School of Business Duke University Durham, NC 27706. October, 1990.

Karatzas, I. and Shreve, S.E. (1991). Brownian Motion and Stochastic Calculus; Springer.

Karatzas, I. and Shreve, S.E. (1994). Graduate Text in Mathematics, Brownian Motion and Stochastic Calculus 2nd ed., Springer-Verlag, New York

Mikosch, T. (1998). Elementary Stochastic Calculus with Finance in View. Singapore: World Scientific Publishing.

Oksendal B. (2003). Stochastic Differential Equations: An Introduction with Applications. Springer, sixth edition

Papoulis A. (1984). Probability, Random Variables and Stochastic Processes, 2nd ed. McGraw-Hill, New York

Sandubete J. E. and Escot, L. (2020). A brief methodological note on chaos theory and its recent applications based on new computer resources. Revista Energeia, Sociedad ibero-Amerocana de Metodologia Economica, Vol. VII, Num. 1, 2020, 53 - 64

Thakur, G. S. M., Bhattacharyya, R. and Mondal, S. S. (2016). Artificial Neural Network Based Model for Forecasting of Inflation in India. *Fuzzy Inf. Eng.* (2016) 8: 87-100.

Zitkovic, G. (2016). Introduction to Stochastic Processes – Lecture Notes. Department of Mathematics, University of Texas, Austin. Retrieved from http://www.ma.utexas-edu/users/gordanz/notes/introduction_to_stochastic_processes