A NEW ROBUST METHOD OF DEALING WITH MULTICOLLINEARITY AND OUTLIERS IN REGRESSION ANALYSIS: SIMULATIONS AND APPLICATIONS

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ABSTRACT

Ordinary Least Squares (OLS) regression is known to be unreliable in the presence of outliers and multicollinearity, leading to biased parameter estimates, inflated standard errors, and reduced predictive accuracy. Limited studies have considered estimators that can co-handle the two problems (multicollinearity and outliers). However, to explore further methods, this study proposes the Robust M-version of the New Biased Based Estimator (M-NBB), designed to handle both multicollinearity and outliers effectively. Theoretical properties of the proposed estimator were established under fundamental conditions and validated through Monte Carlo simulations implemented in R-statistical programming. The simulation study involved 1,000 replications across eight sample sizes and three explanatory variables exhibiting varying degrees of multicollinearity. Additionally, 10% and 20% of the generated observations were contaminated with outliers of various magnitudes under error variances. The performance of the estimators was evaluated using the Mean Squared Error (MSE). The simulation results revealed that the proposed estimator outperformed existing estimators across all conditions. To further validate its effectiveness, the estimator was applied to real-life data. The findings suggest that the M-NBB estimator is a robust and reliable alternative for practitioners dealing with datasets affected by both multicollinearity and outliers.

Keywords: Ordinary Least Squares, Multicollinearity, outliers, Monte Carlo simulation, Estimator

1.0 INTRODUCTION

Linear regression analysis is a widely-used statistical method for modeling the relationship between a dependent variable and one or more independent variables. However, the presence of outliers and multicollinearity can undermine the reliability and accuracy of regression analysis results. Outliers, which are data points that deviate significantly from the rest of the data, can exert undue influence on parameter estimates, leading to biased and unreliable conclusions. Multicollinearity, on the other hand, refers to high correlation among independent variables, which can inflate standard errors and make it difficult to discern the individual effects of predictor variables. Addressing outliers and multicollinearity is crucial for obtaining valid and interpretable regression results, particularly in fields such as finance, economics, healthcare, and engineering, where precise estimation of relationships between variables is essential for decision-making. Traditional least squares estimation method which is also known as Ordinary Least Squares (OLS) are sensitive to outliers and multicollinearity, necessitating the development of robust estimation techniques that can handle these challenges effectively.

As a means of searching for a better robust estimator to simultaneously address multicollinearity and extreme values, this research focuses on developing a new robust M-New Biased Based estimator. This estimator aims to mitigate the problems of multicollinearity and outliers, particularly when outliers are present in the exogenous variable direction, and compare its performance theoretically, via Monte Carlo experiments, and in real-life applications.

2.0 LITERATURE REVIEW

Numerous estimators have been developed in the literature to mitigate the impact of strong correlation among the regressors. Huber (1964) introduced the M-estimator, while Rousseeuw (1984) proposed the Least Trimmed Squares (LTS) estimator as a high-efficiency alternative to the Least Median Squares (LMS) estimator. Additionally, Rousseeuw and Yohai (1984) developed the S-estimator, and Yohai

(1987) introduced the MM-estimator. Rousseeuw and Leroy (1987) created the Least Winsorized Squares (LWS) estimate to reduce the impact of outliers. Rousseeuw and Driessen (1998) introduced the concept of Least Trimmed Mean (LTM). The Ridge estimator was proposed by Hoerl and Kennard (1970) to address multicollinearity. Baye and Parker (1984) introduced the r–k class estimator by combining ridge regression and Principal Component Regression (PCR). Crouse et al. (1995) developed the Unbiased Ridge Regression (URR) estimator. Liu (1993) introduced the Liu estimator, while Kaciranlar and Sakallioglu (2001) combined PCR and the Liu estimator to create the r–d class estimator. Ozkale and Kaciranlar (2007) proposed the two-parameter estimator, which was later integrated by Yang and Chang (2012) with the main component regression estimation. Sakallioglu and Kaciranlar (2008) developed the k–d class estimator. Batah *et al*. (2009) developed the modified r–k class ridge regression (MCRR) estimate. Yang and Chang (2010) combined the URR estimator with PCR to create a new two-parameter estimator. Dorugade (2014) introduced a modified two-parameter estimator. Susanti *et al*. (2014) explored cases where removing data may not be feasible. Alguraibawi et al. (2015) conducted a diagnostic investigation on leverage points in regression analysis. Ajiboye *et al*. (2016) developed robust Ridge and robust Liu estimators by combining robust estimators with Ridge and Liu estimators to address multicollinearity and outliers. Hassan (2017) introduced the Modified Ridge M-estimator. Mansson *et al*. (2018) worked on estimators for the multinomial logit model’s ridge parameter using MSE as a criterion. Lukman *et al*. (2019) proposed a modified ridge-type estimator and later introduced a robust estimator addressing multicollinearity, outliers, and autocorrelation. Ayinde et al. (2020) devised a method utilizing Yang and Chang’s (2012) new Liu-type estimator for estimating model parameters in principal components. Kibria and Lukman (2020) introduced the Kibria-Lukman Estimator (KLE) and the KL estimator. Ahmad and Aslam (2020) proposed the Modified Novel Two-Type Parameter Estimator (MNTPE) for correlated exogenous variables. Lukman et al. (2020) combined the modified ridge-type estimator with the PCR estimator. Dawoud and Kibria (2020) developed the DK estimator. Dawoud and Abonazel (2021) introduced the Robust Dawoud-Kibria estimator. Owolabi et al. (2022) proposed a new Ridge Type estimator. Majeed *et al*. (2022) suggested a robust M-Kibria-Lukman estimator for outliers in the y-direction, while Adejumo *et al*. (2024) considered outliers in the x-direction for the same estimator. Adejumo *et al*. (2023) proposed the Robust M-New Two Parameter (RNTP) estimator for handling multicollinearity and outliers in linear regression models.

Idowu *et al*. (2023) introduced the two-parameter Kibria-Lukman estimator.

2.1 The Classical Linear Regression (CLR) Estimator

The classical linear regression estimator usually refers to as OLS is given as:

$$Y = X\beta_{p\times 1} + \ell \tag{1}$$

where y is an $n\times 1$ vector of observations on the endogenous variables, X is an $n\times p$ design matrix of exogenous variable with full rank, $\beta_{p\times 1}$ is the $p\times 1$vector of unknown regression coefficients, and ℓ is the $n\times 1$ vector random error terms with $E(\ell)=0$ and $E(\ell^T\ell)=\sigma^2 I_n$ such that I_n is an identity matrix of $n\times n$.

If $N=(I,X)$ and $\beta=(\beta_0,\beta_1^T)^T$.

Therefore, equation (1) can still be expressed as:

$$\hat{\beta}_{OLS} = \mu^{-1}X^T y \tag{2}$$

where $\mu = X^T X$

Equation (2) remains Best linear Unbiased Estimator (BLUE) only if all its assumptions are not violated. Some of the causes of the violation are the presence of multicollinearity and Outliers which may be inevitable in the data set. Hence, to curb this effect, Hoerl and Kennard (1970) proposed an alternative by introducing a shrinkage parameter *k* to (2) as given in (3).

$$\hat{\beta}_k = (\mu + kI_p)^{-1} X^T y \tag{3}$$

Such that *k*>0.

However, $(\mu + kI_p)^{-1} X^T y$ has been noted to be sensitive to outliers in the y-direction, therefore Silvapulley (1991) provided a remedial measure by introducing robust M-estimator into ridge estimator called robust M-ridge estimator, defined as:

$$\hat{\beta}_k^M = (\mu + kI_p)^{-1} \mu \hat{\beta}_M \tag{4}$$

where, $\hat{\beta}_M = \min \sum_{i=1}^{n} \rho\left(\frac{y_i - x_{ij}\hat{\beta}}{s}\right)$.

where *s* is a robust estimate of scale expressed as

$$s = \frac{median\left|e_i - median(e_i)\right|}{0.6745}. \tag{4}$$

Some already existing estimators are:

2.1 Liu Regression Estimator

By combining the Stein Estimator with Ridge estimator, Liu (1993) proposed the Liu Estimator follows;

$$\hat{\beta}_L = (\mu + I)^{-1}(\mu + dI)\hat{\beta}_{OLS}. \tag{5}$$

where $d = 1 - \sigma^2 \left[\frac{\sum_{i=1}^{p} \frac{1}{\lambda_i(\lambda_i + 1)}}{\sum_{i=1}^{p} \frac{\alpha_i^2}{(\lambda_i + 1)^2}}\right]$ such that λ_i is the ith Eigen value μ and $\hat{\alpha}_i = Q^T \hat{\beta}$

Robust version of (5) was proposed by Arslan and Billor (2000) defined as;

$$\hat{\beta}_M^d = (\mu + I)^{-1}(\mu + dI)\hat{\beta}_M \tag{6}$$

2.2 Kibria-Lukman Estimator

Kibria and Lukman (2020) developed KL estimator defined as:

$$\hat{\beta}_{KL} = (\mu + kI_p)^{-1}(\mu - kI_p)\hat{\beta}_{OLS} \tag{7}$$

where $k = \frac{\hat{\sigma}^2}{2\hat{\alpha}_i^2 + (\hat{\sigma}^2 / \lambda_i)}$ and $\hat{\beta}_{OLS}$ is the OLS estimator. Also, Majeed *et al*. (2022) proposed Robust-M version of KL estimator as given in equation (8);

$$\hat{\beta}_{KL}^{M} = (\mu + kI_p)^{-1}(\mu - kI_p)\hat{\beta}_M \tag{8}$$

2.3 New Two-Parameter Estimator

As a means of further tackling the problem of multicollinearity Yang and Chang (2010) proposed NTP expressed as:

$$\hat{\beta}_{(k,d)} = (\mu + I_p)^{-1}(\mu + dI_p)\hat{\beta}_k \tag{9}$$

Whereas, Adejumo *et al*. (2023) developed the robust – M version of (9) and defined as:

$$\hat{\beta}_{(k,d)} = (\mu + I_p)^{-1}(\mu + dI_p)\hat{\beta}_k^M \tag{10}$$

2.4 Dawoud-Kibria Estimator

Dawoud and Kibria (2020) proposed the DK estimator, which was noted to outperform others under some conditions and simulation studies. The estimator can be expressed as:

$$\hat{\beta}_{DK} = (\mu + k(1+d)I_p)^{-1}(\mu - k(1+d)I_p)\hat{\beta}_{OLS}$$

Dawoud and Abonazel (2021) proceeded and proposed a robust –M version of DK estimator by introducing $\hat{\beta}_M$ instead of $\hat{\beta}_{OLS}$ used in the DK estimator. They defined the estimator as:

$$\hat{\beta}_{DK} = (\mu + k(1+d)I_p)^{-1}(\mu - k(1+d)I_p)\hat{\beta}_M \tag{11}$$

3.0 METHODOLOGY

3.1 The proposed Robust Estimator

As an alternative to already existing estimators that can mitigate the problem of multicollinearity in linear regression model, Sakallioglu and Kaciranlar (2008) proposed New Biased Based (NBB) estimator defined as:

$$\hat{\beta}_{(k-d)}^{NBB} = (\mu + I)^{-1}(\mu + (k+d)I)\hat{\beta}_k . \tag{12}$$

where $\hat{\beta}_k = (\mu + kI)^{-1} X^T y$

equation (12) can still be expressed as:

$$\hat{\beta}^{NBB}_{(k-d)} = (\mu + I)^{-1}(\mu + (k+d)I)(\mu + kI)^{-1} \mu \hat{\beta}_{OLS} \tag{13}$$

Presence of outliers in the exogenous variable has been proven to affect the NBB estimator hence, it is very important to develop the robust-M version of the NBB estimator through theoretical comparison, simulation experiment and application to real-life under some conditions and theorems. As a result of this, the robust-M version of NBB estimator is hereby proposed and expressed as:

$$\hat{\beta}^{NBB}_{M} = (\mu + I)^{-1}(\mu + (k+d)I)(\mu + kI)^{-1} \mu \hat{\beta}_{M} . \tag{14}$$

where $\hat{\beta}_M$ is the robust M-estimators.

3.1 The Canonical form of Robust M-New Biased Based (M-NBB) Estimator

The canonical form of the general form of equation (1) can be expressed as in (15):

$$y = H\hat{\delta} + U \tag{15}$$

where $H = XZ$ and $\delta = Z^T \beta$, Z is the orthogonal matrix such that $H^T H = Z^T X^T XZ = \lambda = diag(\lambda_1, \lambda_2, ..., \lambda_p)$ where $\lambda_1, \lambda_2, ..., \lambda_p > 0$ are the ordered eigenvalues of $X^T X$.

Let $\hat{\delta}_R$ be the any robust regression estimator of the equations $\sum_{i=1}^{n} \varphi. \left(e_i / j \right) = 0$ and $\sum_{i=1}^{n} \varphi \left(e_i / j \right) s_i = 0$ such that $e_i = y_i - s_i^T \hat{\delta}_R$ where j is a scale parameter and φ is a useful selected function. (Hampel *et al*. (1986) and Huber (1981)).

Some already existing estimators have the following Canonical form

$$\hat{\delta} = \mu^{-1} H^T y , \tag{16}$$

$$\hat{\delta} = \left(\mu + kI_p \right)^{-1} H^T y , \tag{17}$$

$$\hat{\delta}_M (k) = \left(I_p + kH^{-1} \right) \hat{\delta}_m \ , \tag{18}$$

$$\hat{\delta}_M (d) = \left(\mu + I_p \right)^{-1} \left(\mu + dI_p \right) \hat{\delta}_M \ , \tag{19}$$

$$\hat{\delta}_M(KL) = \left(\mu + kI_p\right)^{-1}\left(\mu - kI_p\right)\hat{\delta}_M \quad , \tag{20}$$

$$\hat{\delta}_M(TP) = \left(\mu + kI_p\right)^{-1}\left(\mu + kdI_p\right)\hat{\delta}_M \quad , \tag{21}$$

$$\hat{\delta}_M(DK) = \left(\mu + k(1+d)I_p\right)^{-1}\left(\mu - k(1+d)I_p\right)\hat{\delta}_M \quad , \tag{22}$$

$$\hat{\delta}_M(NTP) = \left(\mu + I_p\right)^{-1}\left(\mu + dI_p\right)\hat{\delta}_M(k) \ , \tag{23}$$

Therefore, the canonical form of the generalized Robust-M NBB estimator of δ is hereby defined as:

$$\hat{\delta}_M(NBB) = (\mu + I)^{-1}\left(\mu + (k+d)I\right)(\mu + kI)^{-1}\mu\hat{\delta}_M \tag{24}$$

3.2 Determination of the Mean Square Error (MSE) for the Robust M-NNB Estimator

Given the MSE of OLS estimator $\hat{\delta}$ which can be expressed as in (25)

$$MSE\left(\hat{\delta}\right) = E(\hat{\delta} - \delta)^T(\hat{\delta} - \delta) \ , \tag{25}$$

$$= tr(\text{cov}(\hat{\delta})) + bias(\hat{\delta})^T bias(\hat{\delta}) \ .$$

For practical purpose (25) can be written as:

$$MSE\left(\hat{\delta}\right) = \sum_{i=1}^{p} \sigma^2 / \lambda_i \ . \tag{26}$$

The MSE of robust regression can be expressed as in equation (26)

$$MSE\left(\hat{\delta}_M\right) = \sum_{i=1}^{p} \omega_{ii} \ . \tag{27}$$

where ω_{ii} is the diagonal element for $\text{cov}(\hat{\delta}_M)$ which is equivalent to ω which is finite.

Hoerl and Kennard (1970) provided the MSE for Ridge estimator $\hat{\delta}(k)$ as follows:

$$MSE\left(\hat{\delta}(k)\right) = \sigma^2 \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + k)^2} + \sum_{i=1}^{p} \frac{k^2\delta_i^2}{(\lambda_i + k)^2} \tag{28}$$

Liu (1993) estimated the MSE for the Liu estimator $\hat{\delta}_d$ as follows:

$$MSE\left(\hat{\delta}_d\right) = \sigma^2 \sum_{i=1}^{p} \frac{(\lambda_i + d)^2}{(\lambda_i + 1)^2 \lambda_i} + (d-1)^2 \sum_{i=1}^{p} \frac{\delta_i^2}{(\lambda_i + 1)^2} \tag{29}$$

MSE for Robust Ridge estimator $\hat{\delta}_R(k)$ is defined as:

$$MSE\left(\hat{\delta}_M(k)\right) = \sum_{i=1}^{p} \frac{\lambda_i^2\omega_{ii} + k^2\hat{\delta}_i^2}{(\lambda_i + k)^2} \tag{30}$$

MSE for robust Liu estimator $\hat{\delta}_M(d)$ denoted as $MSE\left(\hat{\delta}_M(d)\right)$ is given as:

$$MSE\left(\hat{\delta}_M(d)\right) = \sum_{i=1}^{p} \frac{(\lambda_i + d)^2\omega_{ii} + (1-d)^2\delta_i^2}{(\lambda_i + 1)^2} \tag{31}$$

Kibria and Lukman (2020) defined the canonical form of their proposed MSE as follows:

$$MSE\left(\hat{\delta}_M(KL)\right)=\sum_{i=1}^{p}\frac{(\lambda_i-k)^2\omega_{ii}}{\lambda_i(\lambda_i+k)^2}+\sum_{i=1}^{p}\frac{4k^2\delta_i^2}{(\lambda_i+k)^2} \quad (32)$$

MSE of robust Dawoud-Kibria is expressed as:

$$MSE\left(\hat{\delta}_M(DK)\right)=\sum_{i=1}^{p}\frac{(\mu_i-k(1-d))^2\,\omega_{ii}\;{}_{ii}}{(\mu_i+k(1+d))^2}+\sum_{i=p}^{p}\frac{4k^2(1-d)^2\hat{\delta}_M^2}{(\mu_i+k(1+d))^2}, \quad (33)$$

The MSE of the proposed robust New Two-Parameter Estimator Robust M-NTP is as follows:

$$MSE\left(\hat{\delta}_M(NTP)\right)=\sum_{i=1}^{p}\frac{(\mu_i+d)^2\mu_i\,\omega_{ii}}{(\mu_i+1)^2(\mu_i+k)^2}+\sum_{i=p}^{p}\frac{((k+1-d)\mu_i+k)^2\hat{\delta}_M^2}{(\mu_i+1)^2(\mu_i+k)^2}, \quad (34)$$

The MSE of Robust-M New Biased Based (M-NBB) is hereby proposed following the submission of Sakallioglu and Kaciranlar (2008).

$$MSE\left(\delta_M^{NBB}\right)=\sum_{i=1}^{p}\left[\frac{((1-d)\lambda_i+k)^2\delta^2+\lambda_i(\lambda_i+(d+k))^2\omega_{ii}}{(\lambda_i+1)^2(\lambda_i+k)^2}\right]. \quad (35)$$

3.3 Theoretical Comparison of the proposed Robust M-NBB with some other Estimators

Based on the following imposed conditions and theorems, the superiority of the proposed Robust M-NBB estimator over others is hereby ascertained.

Conditions:

(i) The function φ is non-decreasing and skew-symmetric

(ii) $E(e_i)=0$ and $Var(e_i)=1$ finite variance.

(iii) The diagonal element for $\text{cov}(\hat{\delta}_M)$ which is ω_{ii} is finite.

Theorem I:

$$MSE\left(\hat{\delta}_M^{NBB}(NBB)\right)<MSE\left(\hat{\delta}^{NBB}(NBB)\right),\text{ if }\sum_{i=1}^{p}\omega_{ii}<\sum_{i=1}^{p}\sigma^2$$

Proof: The difference between the MSE of Robust M-NBB and NBB Estimators

$$\Delta_{NBB}^{M}=MSE\left(\hat{\delta}_M^{NBB}(NBB)\right)-MSE\left(\hat{\delta}^{NBB}(NBB)\right),$$

$$=\sum_{i=1}^{p}\left[\frac{((1-d)\lambda_i+k)^2\delta^2+\lambda_i(\lambda_i+(d+k))^2\omega_{ii}}{(\lambda_i+1)^2(\lambda_i+k)^2}-\frac{((1-d)\lambda_i+k)^2\delta^2+\lambda_i(\lambda_i+(d+k))^2\sigma^2}{(\lambda_i+1)^2(\lambda_i+k)^2}\right], \quad (36)$$

Further algebraic expression gives (37)

$$=\sum_{i=1}^{p}\left[\frac{(\omega_{ii}-\sigma^2)\left(\lambda_i(\lambda_i+(d+k))^2\right)}{(\lambda_i+1)^2(\lambda_i+k)^2}\right], \quad (37)$$

Therefore, if $(\omega_{ii}-\sigma^2)<0$, then $MSE\left(\hat{\delta}_M^{NBB}(NBB)\right)<MSE\left(\hat{\delta}^{NBB}(NBB)\right)$.

Hence, $\hat{\delta}_M^{NBB}$ estimator is better than $\hat{\delta}^{NBB}$ if and only if $\sum_{i=1}^{p}\omega_{ii}<\sum_{i=1}^{p}\sigma^2$.

Theorem II:

$MSE\left(\hat{\delta}_M^{NBB}(NBB)\right) < MSE\left(\hat{\delta}_M(k)\right)$ if;

$\sum_{i=1}^{p} J_i < \sum_{i=1}^{p} R_i$.

Proof: The difference between the MSE of $\hat{\delta}_M^{NBB}$ and Robust Ridge Estimator

$$\Delta_{RR}^{M-NBB} = MSE\left(\hat{\delta}_M^{NBB}(NBB)\right) - MSE\left(\hat{\delta}_{RR}(k)\right)$$

$$\Delta_{RR}^{M-NBB} = \sum_{i=1}^{p}\left[\frac{((1-d)\lambda_i + k_1)^2\hat{\delta}_i^2 + \lambda_i(\lambda_i + (d+k_1))^2\omega_{ii}}{(\lambda_i+1)^2(\lambda_i+k_1)^2} - \frac{\omega_{ii}\lambda_i + k_2^2\hat{\delta}_i^2}{(\lambda_i+k_2)^2}\right] \tag{38}$$

$$\Delta_{RR}^{M-NBB} = \sum_{i=1}^{p}\left[\frac{q_{11} + q_{12}\,\omega_{ii}}{q_1} - \frac{\omega_{ii}\lambda_i + q_{21}}{q_2}\right] \tag{39}$$

where $q_1 = (\lambda_i+1)^2(\lambda_i+k_1)^2$, $q_2 = (\lambda_i+k_2)^2$, $a_{11} = ((1-d)\lambda_i + k_1)^2\hat{\delta}_i^2$, $q_{12} = \lambda_i(\lambda_i + (d+k_1))^2$ and $q_{21} = k_2^2\hat{\delta}_i^2$ such that k_1, k_2, and are the biasing parameters of Robust M-NBB and Robust Ridge estimators respectively.

$$\Delta_{RR}^{M-NBB} = \sum_{i=1}^{p}\left[\frac{q_{11} + q_{12}\omega_{ii}}{q_1} - \frac{\lambda_i\omega_{ii} + q_{21}}{q_2}\right], \tag{40}$$

$$\Delta_{RR}^{M-NBB} = \sum_{i=1}^{p}\left[\frac{q_{11}q_2 + q_{12}q_2\omega_{ii} - \lambda_i q_1\omega_{ii} - q_1q_{21}}{q_1q_2}\right],$$

$$\Delta_{RR}^{M-NBB} = \sum_{i=1}^{p}\left[\frac{(q_{12}q_2 - \lambda_i q_1)\omega_{ii} - (q_1q_{21} - q_{11}q_2)}{q_1q_2}\right],$$

where $J_i = (q_{12}q_2 - \lambda_i q_1)\omega_{ii}$ and $R_i = (q_1q_{21} - q_{11}q_2)$. Hence, the difference between J_i and R_i is less than zero if $\sum_{i=1}^{p} J_i < \sum_{i=1}^{p} R_i$.

Theorem III

$MSE\left(\hat{\phi}_M(NBB)\right) < MSE\left(\hat{\phi}_M(NTP)\right)$ if;

$\sum_{i=1}^{p} V_i < \sum_{i=1}^{p} W_i$

Proof: The difference between the MSE of Robust M-NBB and Robust M-NTP

$$D_{M-NTP}^{M-NBB} = MSE\left(\hat{\phi}_M(NBB)\right) - MSE\left(\hat{\phi}_M(NTP)\right),$$

$$D_{M-NTP}^{M-NBB} = \sum_{i=1}^{p}\left[\frac{((1-d_1)\lambda_i + k_3)^2\hat{\phi}_i^2 + \lambda_i(\lambda_i + (d_1+k_3))^2\omega_{ii}}{(\lambda_i+1)^2(\lambda_i+k_3)^2} - \frac{(\lambda_i+d_2)^2\omega_{ii} + ((k_4+1-d_2)\lambda_i + k_4)^2\hat{\phi}_i^2}{(\lambda_i+1)^2(\lambda_i+k_4)^2}\right], \tag{41}$$

$$D_{M-NTP}^{M-NBB} = \sum_{i=1}^{p}\left[\frac{f_{11} + f_{12}\,\omega_{ii}}{f_1} - \frac{f_{21}\omega_{ii} + f_{22}}{f_2}\right], \tag{42}$$

where, $f_1 = (\lambda_i + 1)^2 (\lambda_i + k_3)^2$, $f_{11} = ((1-d_1)\lambda_i + k_3)^2 \hat{\phi}_i^2$, $f_{12} = \lambda_i (\lambda_i + (d_1 + k_3))^2$, $f_{21} = (\lambda_i + d_2)^2$, $f_{22} = ((k_4 + 1 - d_2)\lambda_i + k_4)^2 \hat{\phi}_i^2$ such that k_3, k_4, d_1 and d_2 are biasing parameter for both M-NBB and M-NTP respectively.

$$D_{M-NTP}^{M-NBB} = \sum_{i=1}^{p} \left[\frac{f_{11} f_2 + f_{12} f_2 \omega_{ii} - f_1 f_{21} \omega_{ii} - f_1 f_{22}}{f_1 f_2} \right], \quad (43)$$

$$D_{M-NTP}^{M-NBB} = \sum_{i=1}^{p} \left[\frac{(f_{12} f_2 - f_1 f_{21})\omega_{ii} - (f_1 f_{22} - f_{11} f_2)}{f_1 f_2} \right], \quad (44)$$

where $V_i = (f_{12} f_2 - f_1 f_{21})\omega_{ii}$ and $W_i = (f_1 f_{22} - f_{11} f_2)$. Therefore, $V_i - W_i < 0$ if $\sum_{i=1}^{p} V_i < \sum_{i=1}^{p} W_i$.

Theorem VI

$MSE(\hat{\alpha}_M (NBB)) < MSE(\hat{\alpha}_M (MRT))$ if;

$$\sum_{i=1}^{p} D_i < \sum_{i=1}^{p} U_i$$

Proof: The difference between the MSE of Robust M-NBB and Robust M- Modified Ridge Type Estimator (M-MRT)

$$\Theta_{M-MRT}^{M-NBB} = MSE(\hat{\alpha}_M (NBB)) - MSE(\hat{\alpha}_M (MRT)) ,$$

$$\Theta_{M-MRT}^{M-NBB} = \sum_{i=1}^{p} \left[\frac{((1-d_3)\lambda_i + k_5)^2 \hat{\alpha}_i^2 + \lambda_i (\lambda_i + (d_3 + k_5))^2 \omega_{ii}}{(\lambda_i + 1)^2 (\lambda_i + k_5)^2} - \frac{\lambda_i \omega_{ii} + k_6^2 (1 + d_4)^2 \hat{\alpha}_i^2}{(\lambda_i + k_6 (1 + d_4))^2} \right] \quad (45)$$

$$\Theta_{M-MRT}^{M-NBB} = \sum_{i=1}^{p} \left[\frac{n_{11} + n_{12} \omega_{ii}}{n_1} - \frac{\lambda_i \omega_{ii} + n_{21}}{n_2} \right], \quad (46)$$

where $n_1 = (\lambda_i + 1)^2 (\lambda_i + k_5)^2$, $n_2 = (\lambda_i + k_6 (1 + d_4))^2$, $n_{11} = ((1-d_3)\lambda_i + k_5)^2 \hat{\alpha}_i^2$

$n_{12} = \lambda_i (\lambda_i + (d_3 + k_5))^2$ and $n_{21} = k_6^2 (1 + d_4)^2 \hat{\alpha}_i^2$ such that k_5, k_6, d_3 and d_4 are estimated biasing parameters for Robust M-NBB and Robust M-MRT estimators respectively.

$$\Theta_{M-MRT}^{M-NBB} = \sum_{i=1}^{p} \left[\frac{n_{11} n_2 + n_{12} n_2 \omega_{ii} - \lambda_i n_1 \omega_{ii} - n_1 n_{21}}{n_1 n_2} \right], \quad (47)$$

$$\Theta_{M-MRT}^{M-NBB} = \sum_{i=1}^{p} \left[\frac{(n_{12} n_2 - \lambda_i n_1)\omega_{ii} - (n_1 n_{21} - n_{11} n_2)}{n_1 n_2} \right], \quad (48)$$

$$\Theta_{M-MRT}^{M-NBB} = \sum_{i=1}^{p} \left[\frac{D_i - U_i}{n_1 n_2} \right]. \quad (49)$$

where $D_i = (n_{12} n_2 - \lambda_i n_1)\omega_{ii}$ and $U_i = (n_1 n_{21} - n_{11} n_2)$, Thus $D_i - U_i < 0$ if $\sum_{i=1}^{p} D_i < \sum_{i=1}^{p} U_i$.

3.4 Selection of Robust Shrinkage Parameters for the robust M-NBB Estimator.

Assume that $\hat{\delta}_R \sim N(0,1)$. This simply means that $\hat{\delta}_R$ follows a normal distribution with mean zero, variance equal one and covariance matrix equals $A^2\eta^{-1}$. When $\sqrt{n}\left(\hat{\delta}_R^2 - \delta\right) \sim N(0, A^2\eta^{-1})$, hence the assumption holds most especially for practical use, where $A^2 = \dfrac{m_0^2 E\left[\vartheta^2(\varepsilon / m_0)\right]}{E\left[\varphi^{'}(\varepsilon / m_0)\right]^2}$. Such that m_0 is the scale estimate.

Also, the unbiased estimator $\delta_{iM} = \hat{\delta}_{iM}^2$ that is $E(\delta_{iM}) = \hat{\delta}_{\text{iM}}$ and the unbiased estimator $\omega_{ii} = A^2 / \mu_i$ such that $A^2 = \dfrac{c^2(n-p)^{-1}\sum_{i=1}^{n}\left[\omega(\varepsilon_i / c)\right]^2}{\sum_{i=1}^{n}\left[\frac{1}{n}\omega^{'}(\varepsilon_i / c)\right]^2}$.

Biasing parameters for some robust estimators are hereby expressed as follows:

Robust Ridge Regression (RRR) estimator biasing parameter estimated by Lukman *et al*. (2014) is defined as:

$$\hat{k}_{Ridge} = \frac{pA^2}{\sum_{i=1}^{p}\delta_{iM}^2}. \tag{50}$$

Biasing parameter for robust Liu estimator by Kibria and Lukman (2020) and defined as:

$$\hat{d}_{Liu} = 1 - A^2\left[\frac{\sum_{i=1}^{p}\frac{1}{\lambda_i(\lambda_i+1)}}{\sum_{i=1}^{p}\frac{\delta_{iM}^2}{(\lambda_i+1)^2}}\right]. \tag{51}$$

Biasing parameter for Robust Ridge-Liu estimator can be expressed as:

$$\hat{k}_{R-L} = \frac{1}{p}\sum_{i=1}^{p}\frac{A^2}{\hat{\delta}_{iM}^2 - d\left(\frac{A^2}{\lambda_i + \hat{\delta}_{iM}^2}\right)} \tag{52}$$

and

$$\hat{d}_M = \min\left\{\frac{\hat{\delta}_{iM}^2}{\frac{A^2}{\lambda_i} + \hat{\delta}_{iM}^2}\right\}. \tag{53}$$

Robust Dawoud-Kibria (2020) derived the biasing parameter as follows:

$$\hat{k}_M(DK) = \frac{1}{p}\sum_{i=1}^{p}\frac{A}{(1+d_{TP})\left(\frac{A}{\lambda_i}+2\hat{\delta}_{iM}^2\right)} \quad . \tag{54}$$

where $d_{TP}^M = \min\left(\frac{\hat{\delta}_{iM}^2}{\left(\frac{A}{\lambda_i}\right)+\hat{\delta}_{iM}^2}\right)_{i=1}^{p}$

Consequently, taking the partial derivative of (35), the Harmonic mean of the biasing parameters for Robust M-NBB is as follows:

$$\hat{d}_M = \hat{d}_{Rop}^M = \frac{\sum_{i=1}^{p}\frac{\lambda_i\left(\hat{\delta}_{iM}^2-\hat{\sigma}_M^2\right)}{(\lambda_i+1)^2(\lambda_i+k)^2}}{\sum_{i=1}^{p}\frac{\lambda_i\left(\lambda_i\hat{\delta}_{iM}^2+\hat{\sigma}_M^2\right)}{(\lambda_i+1)^2(\lambda_i+k)^2}}, \text{ such that } k=0<k<1 \tag{55}$$

3.5 Simulation Method to Generate Exogenous variables, Response variable and estimate Mean Square Error.

To determine the superiority of the proposed robust estimators over already existing estimators, a Monte Carlo experiment was conducted with the aids of R-statistical programming codes.

Independent variables were generated using the equation given in (56) and used by Lukman, *et al.* (2020).

$$x_{ij} = (1-\gamma^2)^{\frac{1}{2}} z_{ij} + \gamma z_{ip+1}, \; i = 1, 2, \ldots, n, \; j = 1, 2, \ldots, p. \tag{56}$$

where, z_{ij} are independent standard normal pseudo-random numbers and γ means the correlation between any two exogenous variables. In order to exhibits the degrees of correlations between the explanatory variable, five (5) levels of different correlations were considered and they are: 0.8, 0.85, 0.9, 0.95 and 0.99. Meanwhile, the numbers of exogenous variables is p = 3. The variables were expressed in a standardized form. Likewise, the response variable was generated using the following equations:

$$y = \beta_0 + \beta_1 x_2 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \beta_p x_{ip} + \varepsilon_i, \; i = 1, 2, 3, \ldots, p. \tag{57}$$

where ε_i is the error term which is independently and identically normally distributed with mean zero (0) and variance σ^2 that is $\varepsilon_i \sim iidN(0,\sigma^2)$. Zero intercept was assumed for the model in (57) and the values of β was chosen to satisfy the constraints $\beta^T \beta = 1$suggested by Lukman *et al.* (2020). For the sample sizes n= 20, 40, 100 and 250 respectively with standard deviation (1, 5, and 10) the simulation studies was replicated 1000 by following the research of Kaciranlar and Sakallioglu (2001), Chang and Yang (2012), Lukamn *et al*. (2020) and Ayinde, *et al*. (2020**)**. Equation (58) was used to invoke outliers, whereby 10% and 20% of the generated observations were randomly selected and invoked with magnitude of outliers (0, 3, 6, and 9)

$X(i)_{outlier} = f\text{*}Max\,(X_i) + X_i$ (58)

where f is the magnitude of outliers in the x-direction and $X(i)_{outlier}$ is the point of outliers to being replaced. Furthermore, for each replicate, the estimated MSE for each of the estimators was obtained as given in equation (58).

$$MSE(\beta^*) = \frac{1}{1000}\sum_{i=1}^{1000}(\beta^* - \beta)^2 \qquad (59)$$

where β^* is any of the estimators of both existing and proposed estimators.

3.5.1 Description of Data

The data used to justify the simulation results is an economic data used by several authors such as, Ullah *et al*., (2013), Kashif *et al*., (2019) and Lukman and Ayinde, (2018), Adejumo *et al.* (2024) among others.

The regression equation is defined as:

$$y = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + ... + \beta_6 x_6 \qquad (60)$$

where *y* is the total derived employment, x_1 is the gross national product implicit price deflator, x_2 is the gross national product, x_3 is unemployment, x_4 is the size of armed forces, x_5 is the non-institutional population 14 years of age and over and x_6 is the time. Meanwhile, according to (Walker and Birch, 1988) affirmed that the scaled condition number of the data is 43.275. In the same vein, very many of researchers have used this data to identify influential points, such as; (Walker and Birch, 1988, Cook, 1977, Jahufer and Jianbao, 2009, Jahufer, 2013, Yasin and Murat, 2016.

4.0 RESULTS AND DISCUSSION

From the simulation results as it can be seen from Tables 1-3, the following observations were noticed;

(i) As multicollinearity and outliers are simultaneously increasing as expected, OLS performed woefully. That is the MSE of OLS is extremely large.

(ii) MSEs of the estimators considered increase as the error variances (σ^2), levels of multicollinearity (γ) and percentage (Lo), and magnitude (f) of outliers increase.

(iii) As the sample size (n) increases, the MSEs of the estimators decreases.

(iv) When γ >0, L0>0, the percentage of outliers (Lo) increases and sample size (n)

From the simulation results, it was observed that Robust M-NBB estimator outperformed other estimators such as OLS, L-M, among other estimators considered in the study as the two anomalies (Outliers and multicollinearity) occur simultaneously as it has the minimum MSE except in some cases of level multicollinearities. Meanwhile, the values that are bold indicate the minimum MSE.

4.1 Samples of simulation Results

Table 1: Mean Squares Error of estimator when magnitude of outliers are 0, 3 and 9, percentage of outliers are 0.1 and 0.2 with sample size 20

					L0=0.1, f=0						
		OLS	**RE**	**LIU**	**KL**	**NBB**	**R-M**	**L-M**	**KL-M**	**DK-M**	**M-NBB**
, σ=1	γ =0	0.3189	0.2849	0.2779	0.2891	0.3333	0.2843	**0.2778**	0.2887	0.2913	0.5382
	γ =0.8	0.9855	0.424	0.5382	0.3836	1.0354	0.4233	0.5406	0.383	0.6519	**0.3532**
	γ =0.9	1.8789	0.6601	0.7109	0.5703	1.9754	0.659	0.7135	0.5695	1.1104	**0.3714**
	γ =0.95	3.6694	1.1248	0.9988	**0.9365**	3.8588	1.1255	0.9946	0.9394	1.9494	0.9539
	γ =0.99	18	4.8131	3.0792	3.8902	18.934	4.8246	**3.0628**	3.92	8.3318	1289.8

n=20	σ=5	γ =0	7.9715	3.1971	6.6966	3.1457	8.3336	3.1959	6.6965	3.1488	4.6762	**1.1413**
		γ =0.8	24.637	7.5472	13.069	6.6164	25.886	7.574	13.129	6.6571	12.276	**1.7444**
		γ =0.9	46.974	13.427	16.691	11.37	49.384	13.466	16.809	11.446	21.983	**2.9839**
		γ =0.95	91.734	25.055	23.022	20.703	96.47	25.133	23.032	20.87	41.78	**20.141**
		γ =0.99	450	117.37	74.934	94.831	473.35	117.72	**74.783**	95.699	200.71	107606
	σ=10	γ =0	31.886	11.523	26.759	11.269	0.3333	0.2843	**0.2778**	0.2887	0.2913	0.5382
		γ =0.8	98.546	29.606	52.267	26.056	1.0354	0.4233	0.5406	0.383	0.6519	**0.3532**
		γ =0.9	187.89	53.221	66.635	45.125	1.9754	0.659	0.7135	0.5695	1.1104	**0.3714**
		γ =0.95	366.94	99.793	91.891	82.508	3.8588	1.1255	0.9946	**0.9394**	1.9494	0.9539
		γ =0.99	1800	469.12	299.82	379.04	18.934	4.8246	**3.0628**	3.92	8.3318	1289.8
							L0=0.2, f=3					
			OLS	RE	LIU	KL	NBB	R-M	L-M	KL-M	DK-M	M-NBB
	, σ=1	γ =0	0.2306	0.1481	0.1917	0.1435	0.2411	0.1475	0.1918	**0.1429**	0.1903	0.3176
		γ =0.8	0.3478	0.2218	0.2771	0.2137	0.361	0.2205	0.277	**0.2123**	0.2673	0.3859
		γ =0.9	0.4457	0.2264	0.3104	0.2105	0.4695	0.2256	0.3106	**0.2097**	0.3065	0.2178
		γ =0.95	1.3652	0.4585	0.4956	0.3675	1.4087	0.4494	0.4917	**0.3592**	0.5711	0.987
		γ =0.99	5.2744	1.2383	1.2182	0.9158	5.4649	1.2095	1.1746	**0.9063**	1.8943	8800.8
		γ =0	5.7644	1.9155	4.7761	1.7347	6.0271	1.916	4.7776	1.7395	3.1471	**0.6264**
n=20	σ=5	γ =0.8	8.694	3.1123	6.893	2.8998	9.0262	3.0927	6.8935	2.8851	4.7246	**0.9467**
		γ =0.9	11.142	3.3301	7.7227	2.9588	11.738	3.3384	7.7218	2.9892	5.6107	**0.6362**
		γ =0.95	34.131	8.9236	12.341	7.6405	35.218	8.7409	12.253	7.5361	13.802	**1.399**
		γ =0.99	131.86	28.768	30.384	22.549	136.62	28.123	29.247	**22.519**	47.331	25.488
		γ =0	23.058	7.1097	19.101	6.5156	24.108	7.125	19.107	6.5521	12.459	**1.1437**
	σ=10	γ =0.8	34.776	11.628	27.566	10.968	36.105	11.574	27.568	10.931	18.562	**1.9312**
		γ =0.9	44.566	12.828	30.884	11.463	46.95	12.879	30.876	11.607	22.375	**1.3476**
		γ =0.95	136.52	35.103	49.36	30.377	140.87	34.396	49.018	29.988	55.196	**4.1147**
		γ =0.99	527.44	114.66	121.52	90.183	546.49	112.1	116.96	**90.103**	189.32	104.38
							L0=0.1, f=9					
	, σ=1	γ =0	0.2392	0.1537	0.1995	0.1496	0.2485	0.1529	0.1996	**0.1487**	0.1981	0.1882
		γ =0.8	0.5346	0.227	0.3162	0.1962	0.5709	0.2266	0.3159	**0.1959**	0.311	0.2018
		γ =0.9	1.048	0.3405	0.3947	0.2673	1.0833	0.337	0.3968	0.2654	0.4433	**0.2348**
		γ =0.95	1.7802	0.4666	0.4693	**0.3405**	1.896	0.4666	0.4728	0.3449	0.6376	0.4317
		γ =0.99	8.5005	1.7352	1.6842	**1.2537**	9.0336	1.7326	1.7075	1.2869	2.9276	1006.7
		γ =0	5.9799	1.9769	4.9856	1.7979	6.2135	1.9633	4.986	1.7892	3.0496	**0.4623**
n=20	σ=5	γ =0.8	13.366	3.6114	7.8903	3.1238	14.271	3.6305	7.8842	3.1721	6.2462	**0.5854**
		γ =0.9	26.201	6.4945	9.8319	5.4697	27.082	6.4345	9.89	5.5544	10.193	**0.7407**

		γ =0.95	44.504	9.6834	11.67	7.7246	47.401	9.7369	11.763	7.9758	15.717	**0.8802**
		γ =0.99	212.51	41.611	42.1	31.417	225.84	41.606	42.747	32.428	73.209	**15.246**
		γ =0	23.92	7.4332	19.942	6.7652	24.854	7.3865	19.943	6.7431	12.172	**0.7439**
	σ=10	γ =0.8	53.464	14.031	31.562	12.287	57.086	14.12	31.537	12.502	24.988	**1.0096**
		γ =0.9	104.8	25.631	39.316	21.807	108.33	25.398	39.551	22.176	40.743	**1.1618**
		γ =0.95	178.02	38.405	46.655	30.872	189.6	38.631	47.042	31.906	62.887	**1.5249**
		γ =0.99	850.05	166.18	168.4	125.71	903.36	166.18	171.02	129.78	292.85	**10.632**

Table 2: Mean Squares Error of estimator when magnitude of outliers are 0 and 3, percentage of outliers are 0.1 and 0.2 with sample size 40

							L0=0.1, f=0					
			OLS	**RE**	**LIU**	**KL**	**NBB**	**R-M**	**L-M**	**KL-M**	**DK-M**	**M-NBB**
	, σ=1	γ =0	0.1107	0.0994	0.1045	0.0989	0.1176	**0.0994**	0.1045	0.0989	0.1052	0.4982
		γ =0.8	0.2758	0.1735	0.2252	**0.1673**	0.2909	0.1733	0.2252	0.167	0.221	0.2483
		γ =0.9	0.5105	0.2449	0.3462	**0.2258**	0.538	0.245	0.3457	0.2258	0.3658	0.2715
		γ =0.95	0.9816	0.368	0.4946	0.3197	1.034	0.3684	0.496	0.3202	0.6176	**0.2938**
		γ =0.99	4.7535	1.2463	0.8403	0.9911	5.0051	1.2525	0.8579	1.0003	2.2633	**0.6647**
	σ=5	γ =0	2.7687	1.2413	2.6009	1.2036	2.9395	1.2488	2.6009	1.211	1.7289	**0.8431**
n=40		γ =0.8	6.8955	2.2677	5.6039	2.034	7.2735	2.2857	5.6042	2.052	3.657	**0.8204**
		γ =0.9	12.763	3.6553	8.5083	3.1359	13.451	3.6805	8.5136	3.1671	6.1649	**0.803**
		γ =0.95	24.54	6.4318	11.953	5.3155	25.849	6.4772	11.977	5.3758	11.135	**0.9043**
		γ =0.99	118.84	28.265	18.988	22.357	125.13	28.459	19.629	22.671	49.366	**2.5003**
	σ=10	γ =0	11.075	3.9957	10.402	3.9242	11.758	4.0509	10.402	3.9779	6.3229	**1.1347**
		γ =0.8	27.582	8.3445	22.41	7.598	29.094	8.43	22.411	7.6836	13.757	**1.4169**
		γ =0.9	51.054	14.057	34.015	12.162	53.804	14.174	34.044	12.299	23.708	**1.7746**
		γ =0.95	98.16	25.231	47.776	20.933	103.4	25.419	47.866	21.187	43.682	**2.2468**
		γ =0.99	475.35	112.67	75.675	89.151	500.51	113.45	78.338	90.419	196.68	**8.596**
							L0=0.2, f=3					
		γ =0	0.0967	0.0779	0.0905	0.0776	0.1024	0.0779	0.0905	**0.0776**	0.0903	0.2429
		γ =0.8	0.1729	0.1139	0.1474	0.1105	0.1794	0.1137	0.1474	**0.1102**	0.1463	0.2183
	, σ=1	γ =0.9	0.3291	0.1477	0.2172	0.1285	0.344	0.1473	0.2174	**0.1279**	0.2076	0.1723
		γ =0.95	0.4425	0.1883	0.2682	0.1606	0.4583	0.1874	0.2687	**0.1597**	0.2564	0.1791
		γ =0.99	2.0466	0.5046	0.5287	0.3536	2.1136	0.5004	0.5271	0.3581	0.7188	**0.2304**
		γ =0	2.4186	0.8735	2.2625	0.81	2.56	0.8825	2.2625	0.8198	1.3184	**0.4171**
		γ =0.8	4.3233	1.3554	3.6828	1.194	4.4842	1.3518	3.6834	1.1934	2.0649	**0.4368**
	σ=5	γ =0.9	8.2283	1.9998	5.4259	1.6703	8.6008	1.9951	5.4303	1.6847	3.2993	**0.4252**
n=40		γ =0.95	11.062	2.8493	6.7064	2.3623	11.457	2.8452	6.7123	2.4138	4.5842	**0.5007**

		γ =0.99	51.165	10.812	13.126	8.3082	52.84	10.752	13.13	8.6974	17.676	**0.7021**
		γ =0	9.6745	3.0218	9.0501	2.7724	10.24	3.0662	9.0501	2.8209	5.0462	**0.6337**
		γ =0.8	17.293	4.9819	14.731	4.427	17.937	4.97	14.733	4.4357	8.2022	**0.7636**
	σ=10	γ =0.9	32.913	7.6327	21.703	6.5326	34.403	7.6201	21.72	6.6079	13.201	**0.778**
		γ =0.95	44.249	11.018	26.834	9.2926	45.83	11.008	26.853	9.5344	18.32	**0.9672**
		γ =0.99	204.66	42.959	52.476	33.242	211.36	42.73	52.512	34.85	70.667	**1.3727**

Table 3: Mean Squares Error of estimator when magnitude of outliers are 0, 3, 6 and 9, percentage of outliers 0.1 and 0.2 with sample size 100 and 250

						L0=0.1, f=0						
			OLS	**RE**	**LIU**	**KL**	**NBB**	**R-M**	**L-M**	**KL-M**	**DK-M**	**M-NBB**
		γ =0	0.0427	0.0375	0.0408	**0.0374**	0.0479	0.0458	0.0445	0.0458	0.0448	0.1692
	, σ=1	γ =0.8	0.1765	0.1221	0.1603	0.1205	0.1382	0.0935	0.1177	**0.0918**	0.1131	0.2412
		γ =0.9	0.3468	0.1939	0.2864	0.1877	0.2624	0.1416	0.1982	**0.1346**	0.1954	0.24
		γ =0.95	0.6876	0.2788	0.4774	0.2565	0.5111	0.2115	0.3134	**0.1899**	0.3396	0.2629
		γ =0.99	3.4145	0.7929	0.8641	0.6295	2.5017	0.6436	0.6947	0.5071	1.2449	**0.2429**
		γ =0	1.0665	**0.548**	1.0384	0.5621	1.125	0.6693	1.0494	0.676	0.8007	0.7705
	σ=5	γ =0.8	4.4116	1.048	4.0045	0.8793	3.4554	1.0414	1.0494	0.8965	1.7658	**0.5289**
		γ =0.9	8.6696	1.8834	7.1627	1.5078	6.5589	1.7004	1.0494	1.4041	2.95	**0.4893**
n=100		γ =0.95	17.19	3.3166	11.931	2.5012	12.777	3.016	1.0494	2.4142	5.3312	0.4595
		γ =0.99	85.362	15.355	21.599	11.092	62.544	13.452	1.0494	10.447	24.763	**0.4476**
		rho=0	4.2661	1.3857	4.1642	1.4531	4.4999	1.7867	4.1984	1.7889	2.6067	**0.9255**
	σ=10	rho=0.8	17.647	3.6408	16.016	2.9698	13.822	3.7091	11.762	3.2188	6.3105	**0.7486**
		rho=0.9	34.678	6.8541	28.652	5.331	26.236	6.3925	19.812	5.3107	10.897	**0.7975**
		rho=0.95	68.76	12.749	47.721	9.4807	51.108	11.684	31.047	9.3918	20.317	**0.841**
		rho=0.99	341.45	60.908	86.393	43.85	250.17	53.467	65.248	41.556	97.982	**1.0169**
						L0=0.1, f=3						
		γ =0	0.0123	0.0119	0.0122	0.0119	0.0129	**0.0119**	0.0122	**0.0119**	0.0122	0.2309
		γ =0.8	0.025	**0.0225**	0.0243	**0.0225**	0.0264	**0.0225**	0.0243	0.0225	0.0243	0.1361
	, σ=1	γ =0.9	0.0439	0.0355	0.0412	**0.0352**	0.0816	0.0527	0.1304	0.0511	0.0677	0.0946
		γ =0.95	0.0777	0.0527	0.0683	**0.051**	15.54	3.8672	13.037	3.1323	6.7854	0.5696
		γ =0.99	0.3605	0.1244	0.199	0.0948	0.3793	0.125	0.1992	0.0955	0.1847	**0.0547**
		γ =0	0.307	0.191	0.3046	**0.188**	0.3226	0.1915	0.3046	0.1885	0.2523	0.3116
		γ =0.8	0.6243	0.2652	0.6067	**0.2349**	0.6605	0.2666	0.6067	0.2364	0.3616	0.2587
n=250	σ=5	γ =0.9	1.0968	0.3546	1.0305	0.2832	1.1447	0.3532	1.0305	0.282	0.4817	**0.235**
		γ =0.95	1.9413	0.4997	1.7074	0.3689	2.0395	0.505	1.7074	0.3782	0.7268	**0.235**
		γ =0.99	9.0128	1.7309	4.9814	1.2374	9.4817	1.755	4.9894	1.3026	2.8866	**0.2102**
		γ =0	1.2281	0.516	1.2183	0.488	1.2905	0.5203	1.2183	0.4926	0.7491	**0.3943**

		γ =0.8	2.4971	0.7805	2.4269	0.672	2.642	0.7893	2.4269	0.6824	1.1577	**0.3478**
	σ=10	γ =0.9	4.3873	1.1415	4.1222	0.9308	4.5786	1.1381	4.1222	0.9328	1.7715	**0.3559**
		γ =0.95	7.7654	1.7418	6.8294	1.365	8.1579	1.7669	6.8294	1.4155	2.8662	**0.3795**
		γ =0.99	36.051	6.7087	19.932	4.9838	37.927	6.8101	19.962	5.268	11.529	**0.3913**
							L0=0.1, f=6					
		γ =0	0.0121	**0.0117**	0.012	**0.0117**	0.0127	**0.0117**	0.012	**0.0117**	0.012	0.1505
		γ =0.8	0.0246	**0.0221**	0.0239	**0.0221**	0.026	0.0222	0.0239	**0.0221**	0.0239	0.1165
	, σ=1	γ =0.9	0.0434	0.035	0.0407	0.0347	0.0453	0.035	0.0407	**0.0347**	0.0407	0.0964
		γ =0.95	0.0771	0.0522	0.0678	**0.0505**	0.081	0.0523	0.0678	0.0506	0.0675	0.0788
		γ =0.99	0.36	0.124	0.1987	0.0944	0.3787	0.1245	0.1991	0.095	0.1848	**0.0623**
		γ =0	0.3016	0.1846	0.2991	**0.1816**	0.317	0.1851	0.2991	0.1822	0.2476	0.1824
		γ =0.8	0.6151	0.257	0.5976	0.2264	0.6505	0.2582	0.5976	0.2277	0.3533	**0.1519**
n=250	σ=5	γ =0.9	1.0847	0.3436	1.0185	0.2715	1.1315	0.342	1.0185	0.2703	0.4601	**0.1585**
		γ =0.95	1.9287	0.4887	1.6948	0.3573	2.0255	0.4937	1.6948	0.3664	0.705	**0.1593**
		γ =0.99	9.0003	1.7196	4.9713	1.2257	9.4681	1.7432	4.9787	1.2907	2.8674	**0.1489**
		γ =0	1.2063	0.4869	1.1964	0.4601	1.2678	0.4912	1.1964	0.4648	0.7206	**0.2328**
		γ =0.8	2.4605	0.7481	2.3904	0.64	2.6018	0.7561	2.3904	0.6492	1.1127	**0.2326**
	σ=10	γ =0.9	4.3388	1.1002	4.0739	0.8896	4.5259	1.0961	4.0739	0.8917	1.7098	**0.2601**
		γ =0.95	7.715	1.7001	8E+07	1.3233	8.102	1.7239	6.7793	1.3732	2.8122	**0.2671**
		γ =0.99	36.001	6.6648	19.887	4.939	37.872	6.764	19.916	5.2221	11.47	**0.3118**
							L0=0.2, f=9					
		γ =0	0.013	**0.0126**	0.0129	**0.0126**	0.0137	**0.0126**	0.0129	**0.0126**	0.0129	0.0801
		γ =0.8	0.0249	0.0226	0.0243	**0.0225**	0.0262	0.0226	0.0243	**0.0225**	0.0243	0.0464
	, σ=1	γ =0.9	0.0482	0.0384	0.045	**0.038**	0.0509	0.0384	0.045	**0.038**	0.045	0.0542
		γ =0.95	0.075	0.0516	0.0664	**0.0501**	0.0787	0.0516	0.0664	**0.0501**	0.0661	**0.0254**
		γ =0.99	0.346	0.1211	0.1952	0.0929	0.3633	0.1213	0.1955	0.0933	0.1823	**0.0256**
		γ =0	0.3256	0.1946	0.3227	0.1909	0.3422	0.1952	0.3227	0.1915	0.2638	**0.1082**
		γ =0.8	0.6235	0.2616	0.6067	0.2318	0.6558	0.2627	0.6067	0.233	0.3635	**0.0915**
n=250	σ=5	γ =0.9	1.2045	0.382	1.1256	0.302	1.2721	0.385	1.1256	0.3064	0.5193	**0.1277**
		γ =0.95	1.8747	0.4813	1.66	0.3544	1.9677	0.485	1.66	0.3619	0.6886	**0.0873**
		γ =0.99	8.6511	1.6147	4.8791	1.1328	9.0837	1.6324	4.8872	1.1914	2.7223	**0.1026**
		γ =0	1.3024	0.5134	1.2909	0.4825	1.3689	0.5181	1.2909	0.4875	0.7578	**0.1462**
		γ =0.8	2.4941	0.7522	2.4267	0.6395	2.6232	0.7597	2.4267	0.6489	1.1075	**0.1687**
	σ=10	γ =0.9	4.8181	1.2421	4.5024	0.9965	5.0882	1.2592	4.5024	1.0237	1.9753	**0.2292**
		γ =0.95	7.4988	1.6629	6.6399	1.2999	7.8707	1.6817	6.6399	1.3431	2.7457	**0.2151**
		γ =0.99	34.605	6.2387	19.517	4.5665	36.335	6.3139	19.549	4.8237	10.889	**0.2821**

Table 4: Estimated MSEs of the estimators and Regression coefficients

Coefficients	α_{OLS}	α_M	α_M^{RE}	α_M^{LE}	α_M^{KL}	α_M^{RL}	α_M^{MRT}	α_M^{DK}	α_M^{NBB}
β_0	-3482.26	-3639	-0.34014	1044.993	3481.578	-1819.66	-0.22678	-3266.17	-0.17091
β_1	0.015062	-0.0074	-0.05292	106816.2	-0.1209	-0.0174	-0.05289	0.010839	-0.03515
β_2	-0.03582	-0.03625	0.071041	410322.7	0.177902	0.015207	0.071034	-0.02919	0.065094
β_3	-0.0202	-0.02006	-0.00424	336197.8	0.011725	-0.01258	-0.00424	-0.01921	-0.00496
β_4	-0.01033	-0.01061	-0.00573	274094.1	-0.00112	-0.00813	-0.00573	-0.01005	-0.0057
β_5	-0.0511	-0.07188	-0.41393	123068.6	-0.77675	-0.22436	-0.41382	-0.07364	-0.34742
β_6	1.829151	1.911852	0.048579	2042837	-1.73199	0.978939	0.048514	1.718647	0.044866
MSE	792848.7	848073.6	3.715881	4.42E+14	1033.002	3.53E+08	3.711595	747209.3	**3.69521**

4.2 Conclusion

In this research, a Robust M-version of the New Biased-Based (M-NBB) estimator was developed to address multicollinearity and outliers, particularly when outliers are present in the x-direction for linear regression models. The performance of the estimator was evaluated theoretically and through simulation studies using R-statistical codes. The study established certain conditions and theorems under which the estimator demonstrated superiority, with minimum MSE used as the evaluation criterion.

Both the simulation studies and real-life data applications, as shown in Table 4, revealed that the newly developed M-NBB estimator consistently achieved the smallest MSE compared to other estimators considered in the study. This indicates that the M-NBB estimator outperformed several existing methods and is therefore recommended for statistical applications where both multicollinearity and outliers are present in the data.

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