Assessing the Forecast Performance of EGARCH and Prophet Models in Analyzing the Effect of Holidays/Events on Nigeria Stock Price Returns under Normal and GED Assumptions of Innovations

1st U. Abdulaziz, 2nd M. Tasi'u, 3rd A. A. Umar, 4th A. I. Ishaq, 5th A. Usman, & 6th R. O. David

[1,2,3,4,5,6] Department of Statistics, Ahmadu Bello University, Zaria, rabdulazizusman@gmail.com, dagastatistician@gmail.com,aaumar4real@yahoo.com,binishaq05@gmail.com,abubakarusman28@gmail.com, oluwabukunmi360@gmail.com

Abstract

Stock market price returns are greatly affected either positively or negatively by so many factors such as day-of-the-week, holidays/events (Childrens day, Valentine's Day, Id el Kabir, Maulud, Christmas day, Eve day among others). And the effect of such factors on Nigeria Stock market was little or not been assessed and quantified. This paper aimed at studying the effects of holidays/events on the Nigeria Stock market price returns utilizing EGARCH and Prophet models under Normal and Generalized Error Distribution (GED) assumptions of innovations. The results of EGARCH model under Normal assumption of innovations revealed that the holidays that falls on Mondays and Fridays has significant effects on the NSE returns; but the effects are not significant on the other days. Similarly, the results of the Prophet model revealed that all the effect of holidays that falls on Mondays down to Fridays are not significant. On the other hand, the EGARCH model (under GED assumption of innovations) and Prophet model agreed equally as all the holidays effects are not significant in both the cases. With the help of MAE, RMSE, and MASE, EGARCH with GED assumptions of innovations performed better than EGARCH with Normal assumptions of innovations as well as the Prophet model.

Keywords—EGARCH, Facebook prophet, holidays/events effects, GED, NSE.

1. INTRODUCTION

Stock market price returns are greatly affected either positively or negatively by so many factors such as day-of-the-week, holidays (Childrens Day, Valentines Day, Id el Kabir, Maulud, Christmas Day, Eve Day among others). Particularly, stock market volatility of NSE price return described how variance of today's return is influenced or affected by either its single or many past factors, e.g. lags. Market volatility is a crucial part of time series factors that investors are concern about for it provides information and uncertainty of assets return. The changes in variance of assets return gives an investor a hint of their expected return and uncertainty. Ability to specify the best model to capture the volatility of stock price return is of great important as it helps the investors in their risk management and portfolio adjustment (see, Atoi, 2014). Engle (1982) introduced Autoregressive Conditional Heteroskedasticity (ARCH) model in an article Econometrica. He developed it so that economic situations of a given variables volatility will be captured. When this model was developed, many macroeconomist (an expert in macroeconomics; the study of the entire economy in terms of the total amount of goods and services produced, total income earned, and level of employment of productive resources, and the general behavior of prices) were tempted to study inflation volatility (Deacle, 2006). Bollerslev (1986; a graduate student of Engle), and Taylor (1986) proposed the Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model. Many other GARCH family models were established to captured asymmetry characteristics of volatility of stock price return; some of which are EGARCH, TGARCH, IGARCH to mention but few. The models were used in literature review to model the volatility of stock price returns, see Hamilton (2010), Shamiri and Isa (2009) to mention but few. As seen in some literature review in Nigeria, it was stated that modeling the volatility of stock price return with holidays/events effects was not given much attention from most empirical studies on volatility despite the fact that it is one of the most factor to be considered in Nigeria Stock Exchange (NSE). As such, this paper one of the contributions is to fill this gap and provides more empirical studies on the volatility of stock price returns using EGARCH and Prophet models incorporated with holidays/events effects. In addition, many empirical studies in Nigeria do not used Facebook Prophet model (a model where seasonality, trend and holidays functions are incorporated) or consider other error distribution apart from normality assumptions. However, results from many studies has shown that financial series returns are leptokurtic (fat tailed). Hence, this paper also attempt to confirm the leptokurtic (fat-tailed) existence in the literature reviews by using normal and Generalized Error Distributions (GED).

Number of researchers gives evidence that stock returns display volatility clustering, fat-tailed and asymmetric. Changes in variance (volatility) may affected the expected smoothly working system of financial time series returns and economic of a nation (Rajni and Mahendra, 2007; Mollah, 2009). Engle (1982) used an ARCH model to study inflation in the United Kingdom. In estimating the parameters of ARCH model, Engle (1982) realized that Maximum Likelihood Estimation approach is more efficient. Problem of over-parameterizations has been encountered by number of researchers which led to the establishment of GARCH model by Bollerslevs (1986), with the assumption that the volatility has autoregressive structured and positively correlated with its own just past lag and the square of its just past residual. Bollerslevs (1986) gives evidence that GARCH with small order fits best to ARCH of high order. Problems of GARCH models such as assumptions of symmetric and non-negativity of variance may be violated in practical sense. To take care of these problems, number of asymmetric GARCH models were introduced. Some of which are Exponential (EGARCH) proposed by Nelson

(1991), Power ARCH (PARCH) proposed by Ding et al. (1993) and Threshold GARCH (TGARCH) proposed by Zakoian (1994) among others.

The EGARCH (p, q) model allows asymmetric effect of a return to be captured via a coefficient called gamma. It described changes in variance (volatility) in terms of natural logarithm and impose no signs of the volatility and parameter estimates of a series return. According to Nelson (1991), stated that error distribution such as skew students will help to increase kurtosis as well as reduces the serial correlation of the squared observations. Ogum et al. (2005) examined the changes in variance (volatility) of Nigeria's stock market returns series. They used EGARCH model to fit the series so that asymmetric effects can be captured. Their findings revealed the existence of asymmetric effects. Emenike (2010) analyzed the monthly Nigeria Stock Exchange All-shared index using GARCH (1,1) and GJR-GARCH (1,1). The findings revealed that there exist fat-tailed, leverage effect and volatility persistence of the series return. Tasi'u et al. (2014) modeled and forecasted the volatility exchange rate between US (Dollar) and Nigeria (Naira) using GARCH, GJR-GARCH, TGRACH and TS-GARCH models by using daily data over the period, June 2000 to July 2011. Their findings revealed that asymmetric effects exist and significant at 5% level for GJR-GARCH and TGARCH while others did not. They also revealed that TGARCH outperformed other models. The outperformance was realized using symmetric loss functions which are the Mean Absolute Error (MAE), Root Mean Absolute Error (RMAE), Mean Absolute Percentage Error (MAPE) and Theil inequality Coefficient.

Uyaebo et al. (2015) studied daily of all share index series of Nigeria, Kenya, United States, Germany, South Africa and China ranging from February 14, 2000 to February 14, 2013. They fitted the series using asymmetric GARCH models with endogenous break dummy variables under two error assumptions distribution. Their findings indicated the absence of asymmetric effect on Nigeria and Kenya stock market return but presence on the other countries, the volatility of the daily return series of Nigeria and Kenya react to market shock faster than other countries do. Bruno et al. (2020) fitted the series oil price volatility before and after the period of coronavirus crisis using GARCH models. The results revealed that EGARCH (0, 2) best fitted the series. In addition, they revealed that prices are greatly affected, negatively by coronavirus. Oo and Phyu (2020) applied Facebook Prophet model to study temperature in Myintkyina. They used the daily weather data of the years 2010-2017 and trained the forecasting model. They eventually realized that the model was efficient as it provided accurate predictions of temperature in Myitkyina.Oz et al. (2020) investigated the changes in variance in the series returns of BIST 100 index between 01st Jan.2020 to 11th Feb.2021. Furthermore, days of the week, dates of holidays and COVID-19 pandemic effect were considered as dummy variables. In their findings, EGARCH (3, 3) model was realized to be better. Fridays effect, Holidays, and the COVID-19 pandemic caused negative shocks on the volatility (changes in variance) of the series returns and consequently increased volatility of the series returns. They also realized the existence of leverage effect in the series returns. Kazungu et al. (2021) applied both symmetric and asymmetric GARCH models to model changes in variance (volatility) of stock prices for firms listed in the Dar es Salaam Stock Exchange (DSE). Based on the measure of dispersion of the data they examined; standard deviation was high, indicating high fluctuations of series return. They showed that EGARCH (1,1) and GARCH (1,1) best fitted the series than other models. Wang et al. (2021) considered many GARCH models to study the returns and volatility of Bitcoin. The data they used was a closing price of Bitcoin ranging from October 1st, 2013 to July 31st, 2020, where a total observation of 2496 was found in the data set. In particular, they applied GARCH (1,1) and realized clustering characteristics of the series returns and volatility;

the series returns and volatility persistence tends to decrease over time. In order to examine the asymmetric effects, they applied TGARCH and EGARCH models and found out that the leverage effect do not exist.

Aziz et al. (2022) utilized Prophet and hybrid ARIMA models to study crude price oil. They considered SavitzkyGolay smoothing filter to obtain a better performance between the two models. In evaluating the models, they used cross validation using sliding windows on the two models and related their performances using Root Mean Square Error (RMSE) and Mean Absolute Percentage Error (MAPE). They forecasted one-week and one-month interval of the series and realized ARIMA based machine learning approach to be the best model than the Prophet model.

Chen et al. (2023) applied number of GARCH models to model and forecast the series and volatility returns of S and P 500 and related the results between two financial crises, the GFC of 2008 (Global Financial Crises of 2008) and COVID19 financial crises. Based on AIC/BIC, they realized that EGARCH model performed better during the COVID-19 financial crises, while GJR-GARCH was better during the GFC of 2008. Dash (2023) studied AR-GARCH model with day-of-the-week (as dummy variables) for twenty major stocks in Indian banking sector. They used a year data ranging from Apr. 1, 2018 to Mar. 31, 2019. The authors findings showed that the AR-GARCH, to some extend do not showed any consistent day-of-the-week effects across all the twenty considered stocks. The authors equally revealed that the Tuesdays effect has lower returns and higher changes in variance (volatility) compared to Mondays, while Thursdays has higher volatility compared to Mondays. The author recommended more study to be done.

2. METHODOLOGY

The daily stock price return series of NSE for the period 16th December 2009 to 6th February 2019 resulting to 2383 of observations are used. The selected period of the data was due to the only available data in the Central Bank of Nigeria website. Suppose *ηt* and *ηt−1* represent the present and past days stock prices return respectively. The log return series, denoted by *rt* is:

$$r_t = \log\left(\eta_t / \eta_{t-1} \right) \tag{1}$$

The log return series, *rt* will be used as the observing volatility of the Nigeria stock price returns over the period 2009 to 2019.

A. EGARCH(1,1) MODEL

The EGARCH(1,1) is expressed in equation (2) as:

$$\log\left(\sigma_t^{\,2}\right) = \omega_0 + \varpi_1\left(\frac{\left|\varepsilon_{t-1}\right|}{\sigma_{t-1}} - \sqrt{\frac{2}{\pi}}\right) + \gamma_1\left(\frac{\varepsilon_{t-1}}{\sigma_{t-1}}\right) + \varpi_2 \log\left(\sigma_{t-1}^2\right) + \sum_{m=1}^{5} \beta_m D_{mt} \tag{2}$$

where ϖ_1 determines the volatility clustering, ϖ_2 determines the volatility persistence and γ_1 determines whether leverage effect exist or not while σ_t and σ_{t-1} denotes the present and past volatility of the series returns. D_{mt} is a binary variable taken value 1 or 0. When D_{1t} = 1, day *t* is a Monday and 0 otherwise; D_{2t} = 1, day *t* is a Tuesday; D_{3t} = 1, day *t* is a Wednesday D_{4t} = 1, day *t* is a Thursday and $D_{5t} = 1$, day t is Friday.

Note that:

$$\varepsilon_t = \sigma_t z_t \text{ and } z_t \sim N(0,1) \tag{3}$$

B. FACEBOOK PROPHET MODEL

The prophet model incorporate three functions namely trend, seasonality and holidays define as:

$$y(t) = G(t) + S(t) + H(t) + \varepsilon_t. \quad (4)$$

Where ε_t explain the uncaptured factor(s), $y(t)$, $G(t)$, $S(t)$, and $H(t)$, are the series returns, trend function, seasonality function, holidays/events function respectively and defined as follows:

$$G(t) = \frac{c(t)}{1 + e^{\left(-\left(k + \mathbf{a}_j(t)^{\mathrm{T}}\delta\right)\left(t - \left(\mathrm{m} + \mathbf{a}_j(t)^{\mathrm{T}}\gamma_j\right)\right)\right)}}. \quad (5)$$

where c is the carrying capacity, k the growth rate, and m an offset parameter. The carrying capacity is not constant as the number of people involved in a stock market increases, so does the growth ceiling. Hence, we replace the fixed capacity c with a time-varying capacity *c(t).* Also, the growth rate is not constant. As the company/organization offer good interest, it will profoundly alter the rate of growth in a region, so the model must be able to incorporate a varying rate in order to fit historical data.

And $$a_j(t) = \begin{cases} 1, & \text{if } t \geq s \\ 0, & \text{otherwise} \end{cases} \quad (6)$$

$$\gamma_j = \left(s_j - m - \sum_{i<j}\gamma_i\right)\left(1 - \frac{k + \sum_{i<j}\delta i}{k + \sum_{i\leq j}\delta i}\right). \quad (7)$$

$c(t)$ is a carrying capacity varying with time, k the growth rate, m an offset parameter, $\delta \in \mathrm{R}^S$ (a vector rate of adjustment) and δi is the change in rate that occurs at time s_j.

$$s(t) = X(t)\beta = \sum_{n=1}^{N}\left(a_n \cos\left(\frac{2\pi nt}{P}\right) + b_n \sin\left(\frac{2\pi nt}{P}\right)\right) \tag{8}$$

$$H(t) = Z(t)\kappa_i . \tag{9}$$

where κi is a parameter, *Z(t)* is a matrix of regressors and is expressed by:

$$Z(t) = [\mathbf{1}(t \in D_1),\ldots,\mathbf{1}(t \in D_L)] \tag{10}$$

As with seasonality, we use a prior κ ~ N(0,σ2).

C. FORECAST PERFORMANCE MEASURE

Forecast performance measures are usually used to determine among the models which is the best. The measures generally explain in summary the error due to forecasting. Mean Absolute Error (MAE), Mean Absolute Scale Error (MASE) and Root Mean Square Error (RMSE) will be used to ascertained the forecast performances between the EGARCH (1,1) model and Facebook Prophet model. Their mathematical expressions are defined as follows:

$$MAE = \frac{1}{k}\sum_{t=T+1}^{T+k}\left|\sigma^2{}_t - \sigma^2{}_t\right|. \tag{11}$$

$$RMSE = \sqrt{\frac{1}{k}\sum_{t=T+1}^{T+k}\left(\sigma^2{}_t - \sigma^2{}_t\right)^2}. \tag{12}$$

$$MASE = \frac{\frac{1}{j}\sum_j |e_j|}{\frac{1}{k}\sum_{t=T+1}^{T+k}\left|\sigma_t^2 - \hat{\sigma}_t\right|} \tag{13}$$

3. RESULTS AND DISCUSSION

This section basically present and analyzed the data which can be seen below.

	EGARCH(1,1)	
Parameters	**Normal Distribution**	**Generalized Error Distribution**
δ_1	0.00318 (0.01361)	0.00002 (0.76509)
δ_2	0.00231 (0.10915)	0.00003 (0.93057)
δ_3	0.00301 (0.05838)	0.00163 (0.62180)
δ_4	0.00231 (0.12524)	0.00160 (0.56891)
δ_5	-0.00327 (0.01485)	0.00002 (0.80429)
α_1	0.01142 (0.44835)	-0.02321 (0.36150)
β_1	0.92910 (0.00000)	0.88016 (0.00000)
ω	-0.64335 (0.00004)	0.32536 (0.0006)
γ_1	0.32282 (0.00000)	0.43350 (0.00000)
Persistence	0.92910	0.88016

Table 1: PARAMETER ESTIMATE OF EGARCH (1,1) MODEL WITH HOLIDAYS/EVENTS EFFECTS: UNDER NORMAL AND GED ASSUMPTIONS OF INNOVATIONS

Note: the value in the parenthesis are the p-values, while δ_1, δ_2, δ_3, δ_4 and δ_5 are the Mondays, Tuesdays,…, Fridays parameters

According to Table 1, EGARCH under normal assumptions of error term indicated significant of all the holidays that falls on Mondays and Fridays and non for the other days. Additionally, Mondays and Fridays effect coefficients are positive and negative, which means that the return series of Nigeria stock exchange will increase and decrease per unit value of the holidays that falls on Mondays and Fridays respectively. In addition, the findings shows that EGARCH under GED assumptions of innovations indicated non-significant effect of holidays that falls on Mondays down to Fridays.

On the other hand, α_1 is not statistically significant in the two error assumptions of innovations. This indicates the presence of volatility clustering in the models can be ignored even though it exist. β_1, is also statistical significance in the two error assumptions. The persistence of the EGARCH model under the two assumptions is close to 1, implying the effect of past shocks on the current volatility gradually diminished over time.

Parameter	Normal Distribution	Generalized Error Distribution
RMSE	0.009574458	0.009572434
MAE	0.006765499	0.00676252
MASE	0.829656	0.8292907

TABLE 2: FORECAST ACCURACY OF EGARCH (1,1) MODEL: UNDER NORMAL AND GED ASSUMPTIONS OF INNOVATIONS

Based on the forecast accuracy in Table 2, EGARCH under generalized error distribution of innovation (error term) slightly outperformed EGARCH model under normal assumptions of innovations. With this regard, EGARCH (under GED assumptions of innovations) with Facebook Prophet will be compared to see which is better. Hence, table 3.

Parameter	Generalized Error Distribution EGARCH	Normal Distribution Prophet
δ_1	0.00002 (0.76509)	0.037166 (0.6777)
δ_2	0.00003 (0.93057)	0.019149 (0.6418)
δ_3	0.00163 (0.62180)	0.050025 (0.6539)
δ_4	0.00160 (0.56891)	-0.064744 (0.6539)
δ_5	0.00002(0.80429)	-0.066770 (0.6836)
α_1	-0.02321 (0.36150)	
β_1	0.88016 (0.00000)	
ω	0.32536(0.0006)	
γ_1	0.43350(0.00000)	
Persistence	0.88016	

TABLE 3: PARAMETER ESTIMATE OF EGARCH MODEL (UNDER GED) WITH HOLIDAYS/EVENTS EFFECTS WITH FACEBOOK PROPHET MODEL

Note: the value in the parenthesis are the p-values, while δ_1, δ_2, δ_3, δ_4 and δ_5 are the Mondays, Tuesdays,…, Fridays parameters.

Based on Table 3, both EGARCH and Prophet models indicated non-significant effect of all holidays that falls on Mondays down to Fridays at 5% level. Furthermore, the parameters α1, β1, ω and γ1 are as explained in Table 2

Parameter	Generalized Error Distribution: EGARCH	Normal Distribution: Prophet
RMSE	0.009572434	0.009855862
MAE	0.00676252	0.007060411
MASE	0.8292907	0.8658212

TABLE 4: FORECAST ACCURACY OF EGARCH (UNDER GED) MODEL AND PROPHET MODEL

Despite the fact that both EGARCH (1,1) and Prophet models agreed equally of non-significant effect of all holidays that falls on Mondays down to Fridays, Table 4 shows that EGARCH (1,1) model outperformed Prophet model because of the minimum forecast error.

CONCLUSION

Empirically speaking, to two decimal places, the results indicate the EGARCH and Prophet model to have equal performances. With this regard, the EGARCH or Prophet model can be used to forecast the Nigeria stock price returns. On the other hand, if volatility of the series returns is desired, the EGARCH model is appropriate, but if it is the trend and seasonality of the series returns is desired, Prophet model is more appropriate to be used. More so, under the two distributional assumptions of innovations, there exist the leverage and holidays/events effects (which are all positive), while Prophet model revealed Thursdays and Fridays effects to be negative (such as Boxing day, Good Friday), and Mondays down to Wednesdays to be positive (such as Easter Monday, Id el Maulud, etc.). Under Normal assumptions of innovations, all the holidays/events that falls on Mondays (with 0.00318 effect) and Fridays (with -0.00327 effects) are significance (indicating the series returns for Mondays will increase while for the Fridays, the series returns will decrease).

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