SIMULATION STUDY OF PARAMETERS' INFLUENCE OF HESTON'S STOCHASTIC-JUMP MODEL ON VOLATILITY SMILE

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Abstract
This paper aims at presenting Heston's Stochastic-Jump model, then study the parameters influence of the model on volatility smile. Complete derivation of the Heston's Stochastic-Jump model was presented. Simulation studies were conducted and results show that Heston's Stochastic-Jump model addresses the shortcomings of the Black-Scholes because the way the volatility is modelled is more realistic from financial market's point of view compared to the constant volatility assumption since it takes into consideration what is observed in financial markets. Hence, combining jumps and stochastic volatility therefore produces models which are more flexible and that can accurately fit observable market data.

Keywords: Heston's Stochastic Model, Heston's Stochastic-Jump Model, Black-Scholes' Model, Volatility Smile

1 Introduction

The Heston model is one of the most popular stochastic volatility option pricing models which was developed to overcome the shortcoming of Black-Sholes model of having constant volatility but the Heston's model sometimes does not produce good fit to market prices at short maturity. Hence, the quest to have a model that will be better at approximating market prices and produce fit better than Heston's Stochastic model motivated Nwobi, et al. (2019) to combine jump components to the existing Heston's model developed by Heston, (1993) which they called Heston's Stochastic-Jump model. However, Black-Scholes model has been the standard benchmark for option pricing in the financial market but its assumption of constant volatility of returns which predicts a flat implied volatility surface is unrealistic as it is a well-known empirical fact that implied volatility is not constant as a function of strike nor as a function of time to maturity and generally exhibits some skewness commonly referred to as a volatility smile. If we consider options on an underlying asset with different strike prices, then the volatilities implied by their market prices should be the same. They measure the risk for the same underlying asset. In many markets the implied volatilities often represent a "smile" or "skew" instead of a straight line. The "smile" is however reflecting higher implied volatilities for deep in- or out of the money options and lower implied volatilities for at-the-money options.

In this paper, we aim at presenting the Heston's Stochastic-Jump model where jump process is incorporated as done by Nwobi, et al. (2019), then study the parameters influence of Heston's Stochastic-Jump model on volatility smile. Section 2 gives brief description of Volatility smile, Section 3 presents Heston's Stochastic-Jump model and its derivation. Section 4 introduces the parameters influence on volatility smile while section 5 concludes the paper.

2 Volatility Smile

Volatility smile is the pattern we can observe on a graph of implied volatility verses strike price for a given expiry date. It will form an upturned curve similar to the shape of a smile, because in the money and out of the money options are observed to have higher implied volatilities than at the money options. The Black-Scholes model is a mathematical model developed by Fischer Black and Myron Scholes in 1973 to price European options. According Nwobi et al. (2018), Ken, (2022) and Onyegbuchulem et al. (2024), the Black–Scholes call and put values depend on $S, K, r, T, \tau,$ and σ. Of all these parameters, only the asset volatility, σ cannot be observed directly. One approach is to extract the volatility from the observed market data. Knowing a quoted option call value *C*, and based on observed $S, K, r, T,$ and τ, we can find such σ that leads to this value. A σ computed this way is known as an implied volatility (σ is implied by data on the market). To find such σ we can use for example MATLAB function sigma = blsimpv$(S,\ K,\ r,\ \tau,\ C)$. It was mentioned in Hull (2006) that equity options traded in American markets did not show a volatility smile before the Crash of 1987 but began showing one afterwards. Despite the disparities, the Black–Scholes theory continues to be highly regarded by both academics and market traders. It is common among traders for option values to be quoted in terms of volatility rather than price, because the implied volatility tends to be less variable than the option price. (Onyegbuchulem et al, 2020).

Another approach to obtain volatility parameter σ for the Black-Scholes model is to use the historical data. Historical volatility is the realized volatility of a financial instrument over a given time period. Generally, this measure is calculated by determining the average deviation from the average price of a financial instrument in the given time period.

The Black-Scholes model is widely popular due to its simplicity and ease of calculation. One of the implications of this is that the Black-Scholes stochastic differential equation results in a lognormal distribution of the random variable S_t (its log is normally distributed) however, making a strong assumption by treating volatility as being constant when in real market data, fat tails and a high central peak can be observed in the return distribution. The fat tails and highly skewed central peak observed in real market data is illustrated in the Figure 1, extreme events occur more frequently than a model based on normal random variables would predict.

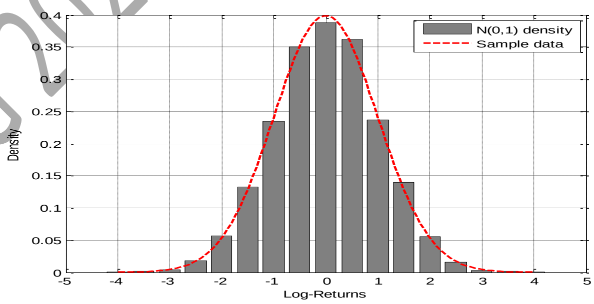

Figure 1: NASDAQ Daily Log-Returns vs. the Normal Distribution

There have been numerous attempts to develop generalizations or alternatives to the lognormal asset price model. Many of these were motivated by the observations of the real market data. One known approach is to allow the volatility to be stochastic, another is to allow the asset to undergo ‘jumps’. We therefore introduce the Heston’s Stochastic-Jump model in the next section which allows volatility to be stochastic and incorporates jump components in asset price.

3.0 Heston's Stochastic-Jump Model

The Heston's Stochastic-Jump Model is given as:

$$dS_t = \left(r - \alpha\mu_{R_t}\right)S_t dt + \sqrt{v_t}S_t dB_1 + R_t S_t dN_t \tag{1}$$

$$dv_t = \kappa\left(\theta - v_t\right)dt + \sigma\sqrt{v_t}\,dB_2 \tag{2}$$

$$c\,\mathrm{ov}\left|dB_1, dB_2\right| = \rho dt,$$

$$\Pr\left[dN_t = 1\right] = \alpha dt$$

where

r is the riskless rate. S_t is the asset price at time t. v_t is the asset price variance at time t.

R_t is the random variable that dictates the percentage jump size of the stock price conditional on the jump occurring, where $\ln\left(1+R_t\right)$ is normally distributed with mean $\ln\left(1+\mu_t\right)-\frac{\delta^2}{2}$

i.e. $\ln\left(1+R_t\right) \sim N\left(\ln\left(1+\mu_{R_t}\right)-\frac{1}{2}\delta^2, \delta^2\right)$

and the variance δ^2, $\left(1+R_t\right)$ has a lognormal distribution:

$$\frac{1}{\left(1+R_t\right)\delta\sqrt{2\pi}}\exp\left(\frac{\left[\ln\left(1+R_t\right)-\left(\ln\left(1+\mu_{R_t}\right)-\frac{\delta^2}{2}\right)\right]^2}{2\delta^2}\right)$$

θ is the long-term variance level, κ is the mean reversion speed, σ is the volatility of variance, ρ is the correlation between the Weiner processes (Brownian motions) B_1 and B_2.

μ_{R_t} is the mean of R_t, δ^2 is the variance of $\ln\left(1+R_t\right)$, α is the annual frequency (intensity) of Poisson process N_t. The three new parameters added to the original Heston's model are μ_{R_t}

, δ^2 and α. The parameters μ_{R_t} and δ^2 determine the distribution of the jumps and the Poisson process is assumed to be independent of the Wiener processes(Brownian motion).

3.1 Derivation of Heston's Stochastic-Jump Partial Differential Equation

Assuming that the stock price and the variance satisfy equations (1) and (2), deriving the Heston-Jump Partial Differential Equation requires forming a riskless portfolio. Setting up a portfolio Π which contains the option being priced with its value denoted by $M = M(S,v,t)$ Δ units of the stock S, φ units of another options $N = N(S,v,t)$ which hedges the volatility.

$$\Pi = M - \Delta S - \varphi N \tag{3}$$

The change in the portfolio in time dt is given by:

$$d\Pi = dM - \Delta dS - \varphi dN \tag{4}$$

We now apply Itô's Lemma to dM and dN in (4) and differentiate with respect to the variables $S, v,$ and t. It should be noted that Cont and Tankov (2004) gives Itô formula for the jump process as :

$$df(X_t,t) = \frac{\partial f(X_t,t)}{\partial t} + b_t \frac{\partial f(X_t,t)}{\partial X} + \frac{1}{2}\sigma^2 \frac{\partial f(X_t,t)}{\partial X^2} + \left[f(X_{t-} + cX_t) + f(cX_{t-}) \right] \tag{5}$$

Applying Cont and Tankov (2004) idea of Itô formula for the jump process in (5) to dM in (4) we have:

$$\begin{aligned} dM = &\frac{\partial M}{\partial t}dt + \frac{\partial M}{\partial S}dS + \frac{\partial M}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2}dt + \frac{1}{2}\sigma^2 v \frac{\partial^2 M}{\partial v^2}dt \\ &+ \sigma v \rho S \frac{\partial^2 M}{\partial v \partial S}dt + \left[M(R_t S,t) - M(S,t) \right] dN_t \end{aligned} \tag{6}$$

The term $\left[M(R_t S,t) - M(S,t) \right] dN_t$ describes the difference in the option value when a jump occurs. Applying Itô's Lemma again to dN and differentiating with respect to the variables $S, v,$ and t, to obtain:

$$dN = \frac{\partial N}{\partial t}dt + \frac{\partial N}{\partial S}dS + \frac{\partial N}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2}dt + \frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2}dt + \sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}dt + \left[N\left(RS,t\right) - N\left(S,t\right)\right]dN_t \tag{7}$$

Inserting equations (6) and (7) into (4), the change in the value of portfolio $d\Pi$ is now written as:

$$d\Pi = \left\{ \begin{array}{l} \left(\begin{array}{l} \frac{\partial M}{\partial t}dt + \frac{\partial M}{\partial S}dS + \frac{\partial M}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2}dt + \frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2}dt \\ +\sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}dt + \left[M\left(R_t S_t,t\right) - M\left(S_t,t\right)\right]dN_t \end{array} \right) - \Delta dS \\ -\varphi\left(\begin{array}{l} \frac{\partial N}{\partial t}dt + \frac{\partial N}{\partial S}dS + \frac{\partial N}{\partial v}dv + \frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2}dt + \frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2}dt \\ +\sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}dt + \left[N\left(R_t S,t\right) - N\left(S,t\right)\right]dN_t \end{array} \right) \end{array} \right\} \tag{8}$$

Rearranging equation (8),so that dt terms for M , dt for N , dS , dv and dN_t terms are grouped together to have

$$d\Pi = \left. \begin{array}{l} \left(\frac{\partial M}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2} + \sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}\right)dt \\ -\varphi\left(\frac{\partial N}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2} + \sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}\right)dt \\ +\left(\frac{\partial M}{\partial S} - \varphi\frac{\partial N}{\partial S} - \Delta\right)dS + \left(\frac{\partial M}{\partial v} - \varphi\frac{\partial N}{\partial v}\right)dv \\ +\left(\left[M\left(R_t S_t,t\right) - M\left(S_t,t\right)\right] - \varphi\left[N\left(R_t S,t\right) - N\left(S,t\right)\right]\right)dN_t \end{array} \right\} \tag{9}$$

The two terms ds and dv in (9) contribute to risk in the portfolio according to Heston (1993). However, for the portfolio to be risk free dS and dv must be eliminated by equating their coefficients to zero. The hedge parameters now become

$$\left. \varphi = \frac{\frac{\partial M}{\partial v}}{\frac{\partial N}{\partial v}}, \qquad \Delta = \frac{\partial M}{\partial S} - \varphi\frac{\partial N}{\partial S} \right\} \tag{10}$$

equation (9) now becomes

$$\left.\begin{aligned} d\Pi \quad = & \left(\frac{\partial M}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2}+\sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}\right)dt \\ & -\varphi\left(\frac{\partial N}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2}+\sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}\right)dt \\ & +\Big(\big[M(R_tS_t,t)-M(S_t,t)\big]-\varphi\big[N(R_tS,t)-N(S,t)\big]\Big)dN_t \end{aligned}\right\} \tag{11}$$

The portfolio should also earn a free risk rate, thus:

$$d\Pi = r\left(M-\Delta S-\varphi N\right)dt \tag{12}$$

Equating the right hand of (11) to right hand side of (12), dividing both side by dt ,

$$\left.\begin{aligned} & \left(\frac{\partial M}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2}+\sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}\right) \\ & -\varphi\left(\frac{\partial N}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2}+\sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}\right) \\ & +\Big(\big[M(R_tS_t,t)-M(S_t,t)\big]-\varphi\big[N(R_tS,t)-N(S,t)\big]\Big)dN_t \end{aligned}\right\} = r\left(M-\Delta S-\varphi N\right) \tag{13}$$

Plugging the values of φ and Δ from (10), we have

$$\begin{aligned} & \left(\frac{\partial M}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 M}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 M}{\partial v^2}+\sigma v\rho S\frac{\partial^2 M}{\partial v\partial S}\right) \\ & -\frac{\partial M/\partial v}{\partial N/\partial v}\left(\frac{\partial N}{\partial t}+\frac{1}{2}vS^2\frac{\partial^2 N}{\partial S^2}+\frac{1}{2}\sigma^2 v\frac{\partial^2 N}{\partial v^2}+\sigma v\rho S\frac{\partial^2 N}{\partial v\partial S}\right) \\ & +\left(\big[M(R_tS_t,t)-M(S_t,t)\big]-\frac{\partial M/\partial v}{\partial N/\partial v}\big[N(R_tS,t)-N(S,t)\big]\right)dN_t \\ & = r\left(M-\frac{\partial M}{\partial S}S+\frac{\partial M/\partial v}{\partial N/\partial v}\left(\frac{\partial N}{\partial S}\right)S-\frac{\partial M/\partial v}{\partial N/\partial v}N\right) \end{aligned} \tag{14}$$

Rearranging (14), such that M terms will be one side and N terms will be in another side, then divide both sides by $\frac{\partial M}{\partial v}$ to obtain:

$$\left. \begin{Bmatrix} \frac{\partial M}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 M}{\partial v^2} + \sigma v \rho S \frac{\partial^2 M}{\partial v \partial S} - rM + rS \frac{\partial M}{\partial S} \\ + \left[M\left(R_t S_t, t\right) - M\left(S_t, t\right) \right] dN_t \end{Bmatrix} \middle/ \frac{\partial M}{\partial v} \right. = \left. \begin{Bmatrix} \frac{\partial N}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 N}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 N}{\partial v^2} + \sigma v \rho S \frac{\partial^2 N}{\partial v \partial S} - rN + rS \frac{\partial N}{\partial S} \\ + \left[N\left(R_t S_t, t\right) - N\left(S_t, t\right) \right] dN_t \end{Bmatrix} \middle/ \frac{\partial N}{\partial v} \right. \quad (15)$$

Taking expectation over the probability distribution of jumps, we obtain

$$\left. \begin{Bmatrix} \frac{\partial M}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 M}{\partial v^2} + \sigma v \rho S \frac{\partial^2 M}{\partial v \partial S} - rM + rS \frac{\partial M}{\partial S} \\ + \alpha E\left[M\left(R_t S_t, t\right) - M\left(S_t, t\right) \right] dN_t \end{Bmatrix} \middle/ \frac{\partial M}{\partial v} \right. = \left. \begin{Bmatrix} \frac{\partial N}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 N}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 N}{\partial v^2} + \sigma v \rho S \frac{\partial^2 N}{\partial v \partial S} - rN + rS \frac{\partial N}{\partial S} \\ + \alpha E\left[N\left(R_t S_t, t\right) - N\left(S_t, t\right) \right] dN_t \end{Bmatrix} \middle/ \frac{\partial N}{\partial v} \right. \quad (16)$$

note that:

$$E\left[M\left(R_t S_t, t\right) - M\left(S_t, t\right) \right] = \int_0^\infty \left[M\left(R_t S_t, t\right) - M\left(S_t, t\right) \right] M\left(R_t\right) dR_t \quad (17)$$

Equation (21) is the expected value of the change in the option price with respect to the jump probability distribution function. Equation (16) now becomes

$$\left. \begin{Bmatrix} \frac{\partial M}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 M}{\partial v^2} + \sigma v \rho S \frac{\partial^2 M}{\partial v \partial S} - rM + rS \frac{\partial M}{\partial S} \\ + \alpha \int_0^\infty \left[M\left(R_t S_t, t\right) - M\left(S_t, t\right) \right] M\left(R\right) dR \end{Bmatrix} \middle/ \frac{\partial M}{\partial v} \right. = \left. \begin{Bmatrix} \frac{\partial N}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 N}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 N}{\partial v^2} + \sigma v \rho S \frac{\partial^2 N}{\partial v \partial S} - rN + rS \frac{\partial N}{\partial S} \\ + \alpha \int_0^\infty \left[N\left(R_t S_t, t\right) - N\left(S_t, t\right) \right] N\left(R_t\right) dR_t \end{Bmatrix} \middle/ \frac{\partial N}{\partial v} \right. \quad (18)$$

The expression in terms of M and that in terms of N in (18) are the same but represent different options. This means that each of the two expressions can be written as a function $M\left(S, v, t\right)$

of $S, v,$ and t. Following Heston (1993), this function can be specified as $M(S,v,t) = -\kappa(\theta - v) + \alpha(S,v,t)$, that is

$$\left\{ \begin{array}{l} \frac{\partial M}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 M}{\partial v^2} + \sigma v \rho S \frac{\partial^2 M}{\partial v \partial S} - rM + rS \frac{\partial M}{\partial S} \\ +\alpha \int_0^\infty \left[M(R_t S_t, t) - M(S_t, t) \right] M(R_t)\, dR_t \end{array} \right\} \Bigg/ \frac{\partial M}{\partial v} \tag{19}$$
$$= -\kappa(\theta - v) + \alpha(S,v,t)$$

Multiplying both sides of (19) by $\frac{dM}{dv}$ and rearranging to obtain

$$\begin{aligned} &\frac{M}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 M}{\partial v^2} + \sigma v \rho S \frac{\partial^2 M}{\partial v \partial S} - rM + rS \frac{\partial M}{\partial S} \\ &+\kappa(\theta - v)\frac{\partial M}{\partial v} - \alpha(S,v,t)\frac{\partial M}{\partial v} + \alpha \int_0^\infty \left[M(R_t S_t, t) - M(S_t, t) \right] M(R_t)\, dR_t = 0 \end{aligned} \tag{20}$$

As written in Heston, the market price of risk is a linear function of the volatility, such that: $\alpha(S,v,t) = \alpha v$. Therefore, equation (20) can be written as

$$\begin{aligned} &\frac{M}{\partial t} + \frac{1}{2} vS^2 \frac{\partial^2 M}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 M}{\partial v^2} + \sigma v \rho S \frac{\partial^2 M}{\partial v \partial S} - rM + rS \frac{\partial M}{\partial S} \\ &+\kappa(\theta - v)\frac{\partial M}{\partial v} - \alpha v\frac{\partial M}{\partial v} + \alpha \int_0^\infty \left[M(R_t S_t, t) - M(S_t, t) \right] M(R_t)\, dR_t = 0 \end{aligned} \tag{21}$$

Equation (21) is the Heston-Jump Partial Differential Equation with the inclusion of jump component which must be satisfied by the value of an option.

4.0 Experiments and Results

4.1 Calibration of Heston Stochastic-Jump Model to Real Market Prices

This section n calibrates the Heston Stochastic-Jump model to real market data obtained from Bloomberg, comprising 576 daily NASDAQ index call option price quotations from September 21, 2024, to November 20, 2024. The data includes information on mid-price, strike price, underlying price, maturity, and corresponding rates. The Table 1 displays the average mid prices $\bar{x}$ and standard deviation S per moneyness group.

Table 1: Average Mid Prices $\bar{x}$ and Standard Deviation S per Moneyness

Moneyness	*Call*		*Put*	
	$\bar{x}$	S	$\bar{x}$	S
Deep ITM	321.41	49.06	357.34	54.56
ITM	213.72	27.25	217.32	31.76
ATM	104.25	41.52	123.14	30.11
ATM	64.73	35.58	87.38	32.33
OTM	43.64	26.47	57.33	29.86
Deep OTM	27.47	21.87	13.78	14.66

The option data was categorized into moneyness groups. The data consists of 19.2% ITM, 35.2% ATM and 45.6% OTM options. In Table 1, the avarage mid-price and its standard deviation per moneyness group are displayed. The average call option prices vary between 27.47 and 241.85, whereas the put options vary between 13.78 and 357.34.

4.2 Implied Volatility Calculation

Table 2 shows the implied volatility calculated using the Black-Scholes model from market data per moneyness group.

Table 2: Average Implied Volatility $\bar{x}$ and Standard Deviation S per Moneyness

Moneyness	*Call*		*Put*	
	$\bar{x}$	S	$\bar{x}$	S
Deep ITM	0.17	0.03	0.26	0.03
ITM	0.17	0.02	0.28	0.04
ATM	0.16	0.04	0.25	0.02
ATM	0.17	0.02	0.25	0.02
OTM	0.18	0.02	0.24	0.18
Deep OTM	0.21	0.03	0.23	0.18

Table 2 shows the implied volatility calculated with the Black-Scholes model from the market data per moneyness group. The implied volatility was calculated for each option and then the average per moneyness group was calculated.

4.3 Calibration Procedure

The calibration procedure involves minimizing the sum of squared percentage errors between model and market implied volatilities using a non-linear least squares procedure.

Let $\sigma_i(t, S_t K)$ denote the market implied volatility of option i on day t and let $\sigma_i^*(t, S_t K)$ denote the model implied volatility of the option i on day t. The Heston Stochastic-Jump model has eight unknown parameters $\Theta = v_0, \theta, \sigma, \rho, \kappa, \mu_j, \sigma_j, \alpha$ (defined in section 3.0) which need to be calibrated, thus:

$$\min \sum_{i=1}^{N} \left[\frac{\sigma_i(t, S_t, K) - \sigma_i^*(t, S_t, K)}{\sigma_i(t, S_t, K)} \right]^2, t = 1, \ldots, T$$

where N denotes the number of options on day t, and T denotes the number of days in the sample.

The MATLAB function lsqnonlin was used to estimate the parameters, with five different sets of initial parameters as shown in Table 3

Table 3: Initial Parameter Estimates

	v_0	θ	σ	ρ	κ	μ_j	σ_j	α
Initial Estimate A	0.30	0.4	0.30	-0.50	1.00	0.5	3	1
Initial Estimate B	0.048	0.48	0.20	-0.36	1.37	-0.03	-0.3	0.05
Initial Estimate C	0.067	0.067	0.20	-0.33	1.34	0.2	-0.12	0.02
Initial Estimate D	0.054	0.054	0.20	-0.31	1.33	0.2	-0.10	0.02
Initial Estimate E	0.041	0.041	0.20	-0.30	1.31	0.2	-0.10	0.02

4 .4 The Hestons' Stochastic-Jump Parameters' Influence on the Implied Volatility

This section analyses and presents the Hestons' Stochastic-Jump parameters' influence on the implied volatility. The following values were used in the simulations:

the asset price $S_t = 1$, the riskless rate $r = 0$, time to maturity T = 1 year, strike prices range from 0.8 to 1.5, the speed of mean reversion $\kappa = 1$, the long-run volatility level $\theta = 0.4$, volatility of volatility $\sigma = 0.3$, the correlation between the price and the volatility processes $\rho = -0.5$, the initial volatility of the asset price $v_0 = 0.4$, the random jump size $\sigma_R = 3$, the mean of jump process $\mu_{R_t} = 0.5$, the annual frequency (intensity) of Poisson process $\alpha = 1$, the variance of jump process $\delta^2 = 0.3$. We will begin our analysis by showing the influence of some of the parameters on the implied volatility across different strikes.

4. 4.1 The Influence of Correlation, ρ and Volatility of Variance, σ Parameters on the Implied Volatility

The Correlation between the price and the volatility processes is denoted by ρ which determines the shape of volatility smile or skew, while the volatility of variance parameter σ controls the kurtosis. When σ is high, the variance process is highly dispersed, so we expect the distribution of returns to have higher kurtosis and fatter tails than when σ is small. Figure 2 shows the effects of varying ρ and σ on the volatility smile.

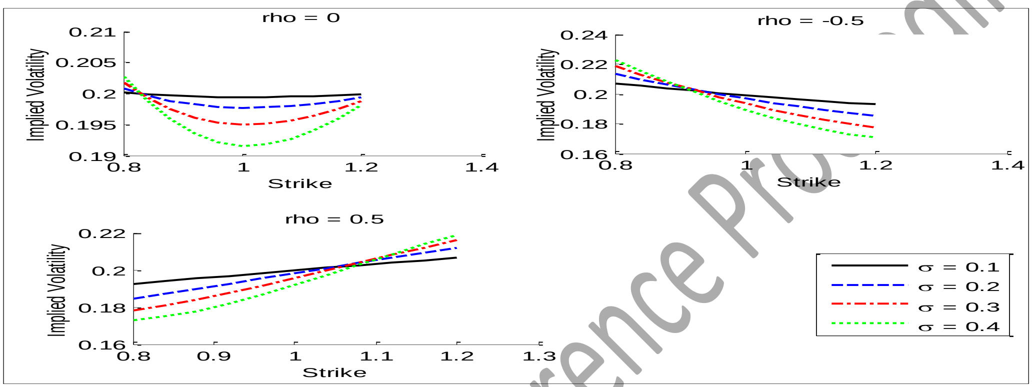

Figure 2: Effects of Varying ρ and σ on the Volatility Smile

Looking at the case where $\rho = 0$, we notice that the implied volatility resembles a smile, rather than a skew, which implies that, when there is no correlation there is no impact on the skewness. In the case where $\rho = -0.5$,it can be observed that the more the strike level increases, the more the implied volatility at expiry. At the money options (ATM) are options where the stock price is equal to the strike price. While out of the money options (OTM) are options where stock price is lower than the strike price and there would decreases. This will result, in the money (ITM) options having high volatility while out of the money (OTM) of the distribution options will have the low volatility. Also, when $\rho = 0.5$, the volatility increases as the strike level increases. However, ITM options will have the low implied volatility. As explained by Heston (1993) in (Onyegbuchulem et al, 2020), positive correlation implies a rise in variance when the stock price rises. Note that in the money (ITM) options are options where the stock price is higher than the strike price and there would be profits be no profits at expiry.

4. 4.2 Influence of Volatility and its Long-Term Level on Implied Volatility

The volatility and its long-term level i.e. θ and v_t have a similar influence on the implied volatility smile. As can be seen from the Figure 3, different levels for θ and v_t barely change the shape of the smile. However, when there are higher values of θ and v_t there will be an upward shift in the smile.

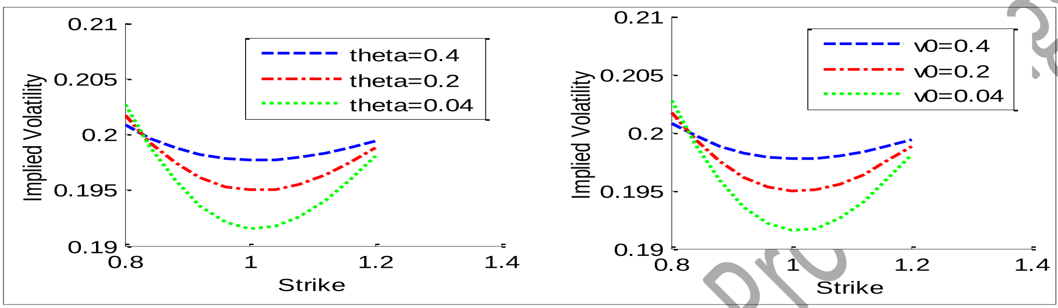

Figure 3: Effect of Varying θ and v_t on the Volatility Smile

4. 4.3 Influence of the Speed of Mean-Reversion, κ on Implied Volatility

Another parameter to be analysed is the speed of mean-reversion κ, which determines the degree of volatility clustering. Figure 4 illustrates the effects of different levels of κ on the implied volatility.

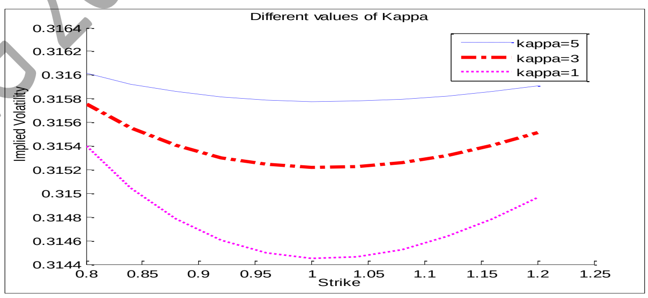

Figure 4: Effect of Varying κ on the Volatility Smile

The smile is more prominent when the value of κ is low and gets flatter as κ increases.

4. 4.4 Influence of Jump Parameter μ_{R_t} on Implied Volatility

The jump parameter μ_{R_t} affects the skewness of the distribution of price returns, positive skewed distributions are as a result of positive values of μ_{R_t} while negative values result in the opposite effect. The effect of changing the skewness of the distribution also impacts on the shape of the implied volatility surface, such that the volatility increases as the asset return increases as illustrated in Figure 5.

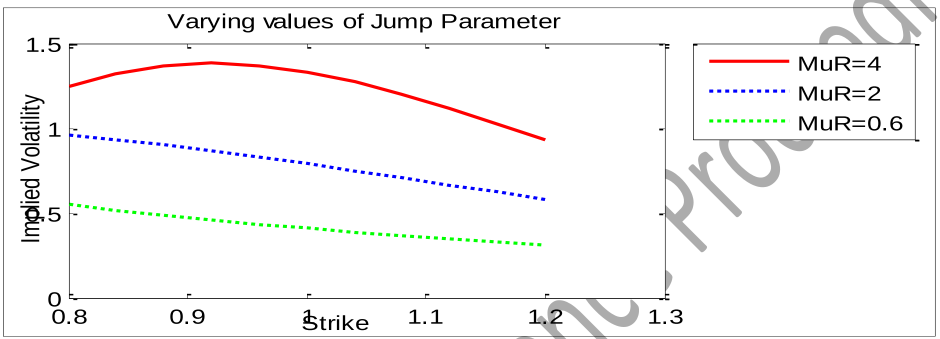

Figure 5: Effect of Varying μ_{R_t} on the Volatility Smile

The variance of the random jump size δ^2 influences the kurtosis of the distribution of returns. Higher values of δ^2 result to a higher variance in the size of the jumps produced by the price process. Finally, the intensity parameter α of the Poisson process N_t determines the frequency of jump occurrence. Higher values of α result to higher number of jumps in the price process and consequently to a higher overall volatility. However, α affects the kurtosis of the distribution of returns similar to δ^2.

5 Conclusion

Having made this analysis on the implied volatility, we can now draw the conclusion that Heston's Stochastic-Jump model addresses the shortcomings of the Black-Scholes because the way the volatility is modelled is more realistic from a financial market's point of view compared to the constant volatility assumption since it takes into consideration what is observed in financial markets, namely the volatility's mean reversion, the leverage effect, volatility clustering and the negative correlation between stock returns and volatility. Hence, combining jumps and stochastic volatility therefore produces models which are more flexible and that can accurately fit observable market data.

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