A BURR X-PERKS DISTRIBUTION: PROPERTIES AND APPLICATIONS

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ABSTRACT

Probability distributions and associated properties have been used extensively over the years for modelling real life problems, however, most conventional distributions do not adequately analyze many of the skewed real-life datasets and therefore the need for compound or extended probability models. This paper presents a study on a new extension of the Perks distribution by adding one shape parameter to the conventional Perks distribution using the Burr X-G family of distributions. This study has derived and investigated some statistical properties of the Burr X-Perks distribution such as moments, moment generating function, the characteristics function, quantile function, survival function and hazard function. Some plots of the distribution and the reliability functions were generated and interpreted appropriately. The results from the curves show that the distribution is skewed with many shapes depending on the values of the parameters. The plot of the survival and hazard functions shows that the distribution can be used to model time-dependent events, where probability of survival decreases with time, while that of failure increases with time. The parameters of the new model have been estimated using the method of maximum likelihood estimation. The paper evaluated the performance of the proposed Burr X-Perks distribution using two real life datasets and the results revealed that the proposed Burr X-Perks distribution fits the two real life datasets better than the three other distributions considered in this study.

Keywords: Burr X-G family, Perks distribution, Properties, Estimation and application.

1 Introduction

The Burr type X is among the twelve different forms of cumulative distribution functions introduced by Burr (1942) for modelling life time data. Among these twelve distribution functions, Burr type X received considerable attention where several authors have attempted to increase its flexibility (Faton *et al*., 2016). The two parameters Burr type X has several types of distributions like Rayleigh distribution when (θ=1) and Burr type X distribution with one parameter when (λ=1). This particular skewed distribution has played an important role in reliability studies and also can be used quite effectively in modeling and analyzing lifetime data of random phenomena, health, agriculture and biology. Several aspects of the one-parameter ($\lambda = 1$) Burr-Type X distribution have been studied by Sartawi and Abu-Salih (1991), Jaheen (1995, 1996), Ahmad et al. (1997), Raqab (1998) and Surles and Padgett (1998). Recently Surles and Padgett (2001) proposed and observed that the Burr-Type X distribution can be used quite effectively in modelling strength data and also modelling general lifetime data.

The Perks distribution was developed by Perks (1932). There are many applications of the Perks distribution most especially in the field of actuarial science. Haberman and Renshaw (2011) and Richards (2008) found that the Perks distribution has a good fit to pensioner mortality data and appropriate for modelling parametric mortality projection. Chaudhary and Kumar (2013) also did Markov Chain Monte Carlo (MCMC) simulation study for the parameter estimates of the Perks distribution using complete sample. The cumulative distribution function (CDF) and probability density function (PDF) of the Perks distribution are respectively defined as:

$$G(x) = 1 - \frac{1+\alpha}{1+\alpha e^{\beta x}} \tag{1}$$

and

$$g(x) = \alpha\beta e^{\beta x} \frac{1+\alpha}{\left(1+\alpha e^{\beta x}\right)^2} \tag{2}$$

where $x, \alpha, \beta > 0$, and α and β are the scale and shape parameters of the Perks distribution respectively while X is the random variable.

The Perks distribution like many other standard probability distributions is useful for describing real life events, however most of these standard distributions are not able to analyze some heavily skewed datasets as expected. Due to this limitation, many families of distributions useful for extending standard distributions have been developed, and some of the recent ones include the Burr X-G family of distributions by Yousof *et al*. (2017), the truncated Burr X-G family of distributions by Bantan *et al*. (2021), the flexible Burr X-G family of distribution by Al-Babtain *et al*. (2021), the odd Perks-G class of distributions by Elbatal et al. (2022), the Marshall-Olkin-odd power generalized Weibull-G family of distributions by Chipepa *et al*. (2022), the shifted exponential-G family of distributions by Eghwerido *et al*. (2022), an odd Chen-G family of distributions by Anzagra *et al*. (2022), a new sine family of generalized distributions by Benchiha *et al*. (2023), a novel bivariate Lomax-G family of distributions Fayomi *et al*. (2023), and a new Frechet-G family of continuous probability distributions by Ieren *et al*. (2024).

Using some of these families and methods, some researchers have developed extensions of the Perks distribution such as the exponentiated Perks distribution by Singh and Choudhary (2017), the Kumaraswamy-Perks distribution by Oguntunde *et al*. (2018) and the Chen-Perks distribution by Mendez-Gonzalez *et al*. (2023).

Therefore, the goal of this paper is to derive a new extension of the Perks distribution using the Burr X-G family of distributions.

The rest of this paper is structured as follows: the new model, its reliability functions and plots are given in section 2. A simplification of the PDF is presented in section 3. Section 4 presents the derivation and study of some properties of the BXPD such as quantile function, moments and generating functions. The estimation of parameters using maximum likelihood estimation (MLE) is contained in section 5 An application of the BXPD to some real-life datasets is done in section 6 and the conclusion of the study is given in section 7.

2. The Burr X-Perks Distribution (BXPD)

The cumulative distribution function (*cdf*) and the corresponding probability density function (pdf) of the Burr X-G family of distributions according to Yousof *et al*. (2017) are defined respectively as

$$F(x)=\left\{1-\exp\left\{-\left[\frac{G(x)}{1-G(x)}\right]^2\right\}\right\}^{\theta} \tag{3}$$

and

$$f(x)=\frac{2\theta g(x)G(x)}{\left[1-G(x)\right]^3}\exp\left\{-\left[\frac{G(x)}{1-G(x)}\right]^2\right\}\left(1-\exp\left\{-\left[\frac{G(x)}{1-G(x)}\right]^2\right\}\right)^{\theta-1} \tag{4}$$

where $g(x)$ and $G(x)$ are the pdf and cdf of any continuous distribution to be modified respectively and $\theta>0$ is the one extra shape parameter of the Burr X-G family of distribution.
Consequently, the cumulative distribution function (cdf) and the probability density function (pdf) of the Burr X-Perks distribution (BXPD) with parameters α, β *and* θ are defined respectively as:

$$F(x)=\left(1-\mathrm{e}^{-\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^2}\right)^{\theta} \tag{5}$$

and

$$f(x)=2\alpha\beta\theta\mathrm{e}^{\beta x}\frac{\alpha\left(e^{\beta x}-1\right)}{\left(1+\alpha\right)^2}\mathrm{e}^{-\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^2}\left(1-\mathrm{e}^{-\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^2}\right)^{\theta-1} \tag{6}$$

where $x,\alpha,\beta,\theta > 0$, α is a scale parameter and β and θ are shape parameters of the BXPD respectively.
Using equations (5) and (6) above, the Survival function (SF) and the hazard function (HF) of the BXPD are respectively obtained as:

$$S(x) = 1 - F\left(x\right) = 1 - \left(1 - \mathrm{e}^{-\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^2}\right)^{\theta} \tag{7}$$

and

$$h(x) = \frac{f\left(x\right)}{S\left(x\right)} = \frac{2\alpha^2\beta\theta\mathrm{e}^{\beta x}\left(e^{\beta x}-1\right)}{\left(1+\alpha\right)^2\mathrm{e}^{\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^2}}\left(1 - \mathrm{e}^{-\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^2}\right)^{\theta-1}\left\{1 - \left(1 - \mathrm{e}^{-\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^2}\right)^{\theta}\right\}^{-1} \tag{8}$$

where $x,\alpha,\beta,\theta > 0$.

The PDF, CDF, SF and HF of the BXPD using some parameter values are presented in the figure below:

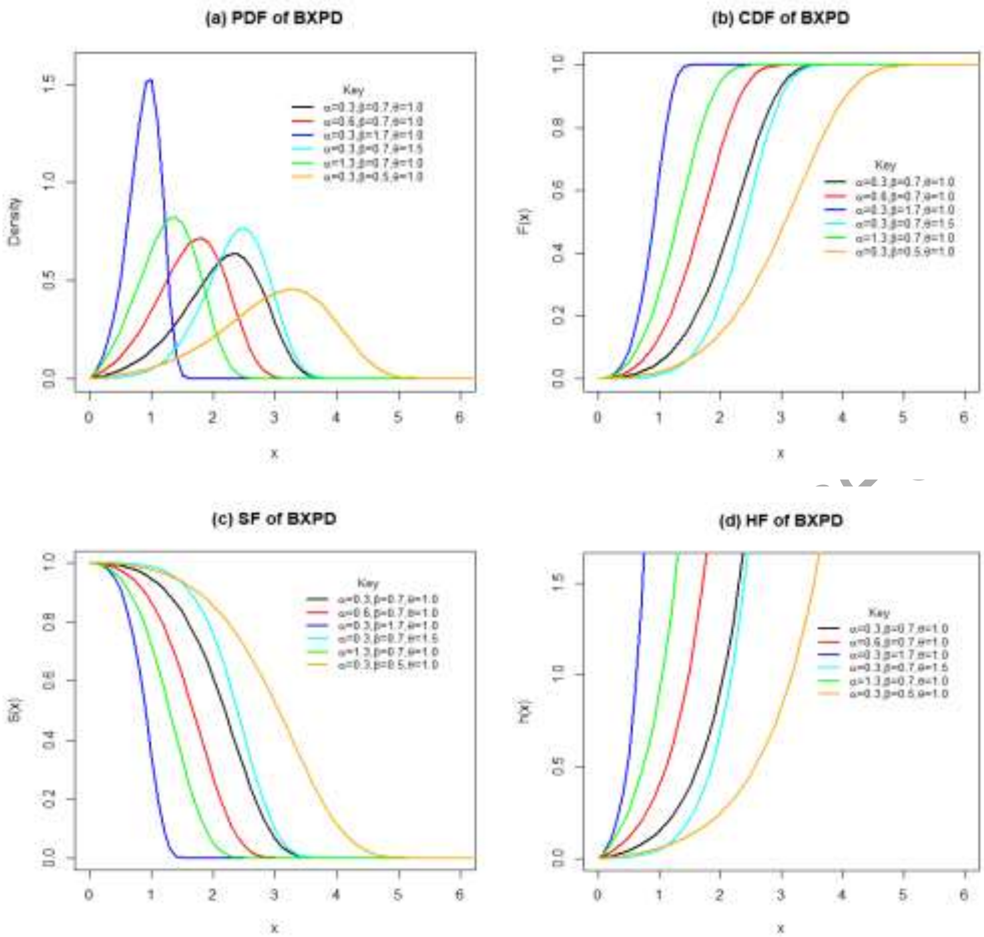

Figure. 1: PDF, CDF, SF and HF of the BXPD for different values of the parameters.

The curves of the PDF and HF of the BXPD show that the BXPD is flexible and its shape varies depending on the values of the parameters. The curves of CDF and SF converge to one as expected which also confirms that the proposed BXPD is a valid probability distribution.

3.0 Simplification of the PDF of Burr X-Perks Distribution (BXPD).

This section presents a simplification of the PDF of the BXPD. Recall that the pdf of the BXPD is given as:

$$f(x) = 2\alpha\beta\theta \mathrm{e}^{\beta x} \frac{\alpha\left(e^{\beta x}-1\right)}{(1+\alpha)^2} \mathrm{e}^{-\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^2} \left(1-\mathrm{e}^{-\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^2}\right)^{\theta-1} \quad (9)$$

Using binomial expansion on the last term in (9) gives:

$$\left(1-\mathrm{e}^{-\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^{2}}\right)^{\theta-1}=\sum_{k=0}^{\infty}(-1)^{k}\binom{\theta-1}{k}\mathrm{e}^{-k\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^{2}} \tag{10}$$

By substituting the result in (10) above and simplifying, the pdf becomes:

$$f(x)=\sum_{k=0}^{\infty}(-1)^{k}\binom{\theta-1}{k}2\alpha\beta\theta\mathrm{e}^{\beta x}\frac{\alpha\left(e^{\beta x}-1\right)}{(1+\alpha)^{2}}\mathrm{e}^{-(k+1)\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^{2}} \tag{11}$$

Recall that according to Taylor series expansion, $\exp(-z)=\sum_{i=0}^{\infty}\frac{(-1)^{i}z^{i}}{i!}$, this implies that the exponential term in pdf of the BXPD in (11) can be expressed as:

$$\mathrm{e}^{-(k+1)\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^{2}}=\sum_{l=0}^{\infty}\frac{(-1)^{l}(k+1)^{l}}{l!}\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^{2l} \tag{12}$$

Also, substituting the result in (12) above and simplifying, the pdf in (12) becomes:

$$f(x)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\binom{\theta-1}{k}\frac{(-1)^{k+3l+1}(k+1)^{l}}{l!(1+\alpha)^{2(l+1)}}2\beta\theta\alpha^{2(l+1)}\mathrm{e}^{\beta x}\left[1-e^{\beta x}\right]^{2l+1} \tag{13}$$

Using binomial expansion, the last term above can be expressed as:

$$\left[1-e^{\beta x}\right]^{2l+1}=\sum_{m=0}^{\infty}(-1)^{m}\binom{2l+1}{m}e^{m\beta x} \tag{14}$$

Again, substituting the result in (14) above and simplifying, the pdf in (13) becomes:

$$f\left(x\right)=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\theta-1}{k}\binom{2l+1}{m}\frac{2\beta\theta\left(-1\right)^{k+3l+m+1}\left(k+1\right)^{l}}{l!\left(1+\alpha\right)^{2(l+1)}\alpha^{-2(l+1)}}e^{\beta(m+1)x} \tag{15}$$

Therefore, equation (15) is the simplified version of the pdf of the BXPD and will be used to derive some properties of the distribution subsequently in section 4.

4. Statistical Properties of the BXPD

In this section, some important statistical properties of the BXPD are derived and discussed as presented.

4.1 Quantile Function, Median and Simulation

According to Hyndman and Fan (1996), the quantile function for any distribution is defined in the form $Q\left(u\right)=F^{-1}\left(u\right)$ where $Q\left(u\right)$ is the quantile function of $F(x)$ for $0<u<1$

To derive the quantile function of the BXPD, the cdf of the BXPD is considered and inverted according to the above definition as follows:

$$F\left(x\right)=\left(1-\mathrm{e}^{-\left[\frac{\alpha\left(e^{\beta x}-1\right)}{1+\alpha}\right]^{2}}\right)^{\theta}=u \tag{16}$$

Simplifying Equation (16) above gives:

$$Q\left(u\right)=\frac{1}{\beta}\ln\left\{1+\frac{1+\alpha}{\alpha}\left[-\ln\left(1-u^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]\right\} \tag{17}$$

Using Equation (17), the median of X from the BXPD is simply obtained by setting $u=0.5$ as follows:

$$Median=\frac{1}{\beta}\ln\left\{1+\frac{1+\alpha}{\alpha}\left[-\ln\left(1-\left(0.5\right)^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]\right\} \tag{18}$$

Similarly, random numbers can be simulated from the BXPD by setting $Q\left(u\right)=X$ as follows:

$$X=\frac{1}{\beta}\ln\left\{1+\frac{1+\alpha}{\alpha}\left[-\ln\left(1-u^{\frac{1}{\theta}}\right)^{\frac{1}{2}}\right]\right\} \tag{19}$$

Also, Kennedy and Keeping (1962) defined the Bowley's measure of skewness based on quartiles as:

$$SK=\frac{Q\left(3/4\right)-2Q\left(1/2\right)+Q\left(1/4\right)}{Q\left(3/4\right)-Q\left(1/4\right)} \tag{20}$$

Relatedly, Moors (1988) presented the Moors' kurtosis based on octiles as:

$$KT = \frac{Q\left(7/8\right) - Q\left(5/8\right) - Q\left(3/8\right) + \left(1/8\right)}{Q\left(6/8\right) - Q\left(1/8\right)} \tag{21}$$

where $Q(.)$ is calculated by using the quantile function from Equation (17).

4.2 Moments and Generating Functions

Let X denote a continuous random variable the *n*^th^ moment of X is given by;

$$\mu_n^{'} = E\left(X^n\right) = \int_0^\infty x^n f\left(x\right)dx \tag{22}$$

Substituting the simplified pdf of the BXPD in equation (15) into equation (22) gives the following;

$$\mu_n^{'} = E\left(X^n\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\theta-1}{k}\binom{2l+1}{m}\frac{2\beta\theta\left(-1\right)^{k+3l+m+1}\left(k+1\right)^l}{l!\left(1+\alpha\right)^{2(l+1)}\alpha^{-2(l+1)}}\int_0^\infty x^n \mathrm{e}^{\beta(m+1)x}dx \tag{23}$$

Making use of integration by substitution method in equation (23) and simplifying yields the following result:

$$\mu_n^{'} = E\left(X^n\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\theta-1}{k}\binom{2l+1}{m}\frac{2\beta\theta\left(-1\right)^{k+3l+m+1}\left(k+1\right)^l}{l!\left(1+\alpha\right)^{2(l+1)}\alpha^{-2(l+1)}}\left(\frac{-1}{\beta\left(m+1\right)}\right)^{n+1}\int_0^\infty u^n \mathrm{e}^{-u}du \tag{24}$$

Recall that $\int_0^\infty t^{k-1}e^{-t}dt = \Gamma\left(k\right)$ and that $\int_0^\infty t^k e^{-t}dt = \int_0^\infty t^{k+1-1}e^{-t}dt = \Gamma\left(k+1\right)$

Considering the statement above, the *n*^th^ ordinary moment of X for the BXPD is obtained as:

$$\mu_n^{'} = E\left(X^n\right) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\binom{\theta-1}{k}\binom{2l+1}{m}\frac{2\beta\theta\left(-1\right)^{k+3l+m+n+2}\left(k+1\right)^l\Gamma\left(n+1\right)}{l!\left(1+\alpha\right)^{2(l+1)}\alpha^{-2(l+1)}\left[\beta\left(m+1\right)\right]^{n+1}} \tag{25}$$

Hence equation (25) is the nth ordinary moment of X for the BXPD and is useful for computing the mean ($\mu_1^{'}$), variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK).

The nth ordinary moment of X for the BXPD can also be used to derive the moment generating function of a random variable X based on power series expansion as follows:

$$M_X\left(t\right) = E\left[\mathrm{e}^{tx}\right] = E\left[\sum_{n=0}^{\infty}\frac{\left(tx\right)^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{t^n}{n!}\int_0^\infty x^n f\left(x\right)dx = \sum_{n=0}^{\infty}\frac{t^n}{n!}E\left(X^n\right) \tag{30}$$

Substituting equation (25) into equation (30) and simplifying, the moment generating function of the BXPD is obtained as:

$$M_X(t) = E\left[e^{tx}\right] = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{t^n}{n!}\binom{\theta-1}{k}\binom{2l+1}{m}\frac{2\beta\theta(-1)^{k+3l+m+n+2}(k+1)^l\,\Gamma(n+1)}{l!(1+\alpha)^{2(l+1)}\alpha^{-2(l+1)}\left[\beta(m+1)\right]^{n+1}} \quad (31)$$

Similarly, the characteristics function of the BXPD can be obtained based on the nth ordinary moment using power series expansion as follows:

$$\phi_X(t) = E\left[e^{itx}\right] = E\left[\sum_{n=0}^{\infty}\frac{(itx)^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\int_0^{\infty}x^n f(x)\,dx = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}E\left(X^n\right) \quad (32)$$

Again, substituting for $E\left(X^n\right)$ in equation (32) and simplifying, the characteristic function of the BXPD is determined as:

$$\phi_X(t) = E\left[e^{itx}\right] = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(it)^n}{n!}\binom{\theta-1}{k}\binom{2l+1}{m}\frac{2\beta\theta(-1)^{k+3l+m+n+2}(k+1)^l\,\Gamma(n+1)}{l!(1+\alpha)^{2(l+1)}\alpha^{-2(l+1)}\left[\beta(m+1)\right]^{n+1}} \quad (33)$$

5. Point Estimation of the unknown Parameters of the BXPD

Let $X_1, X_2, \ldots\ldots\ldots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the BXPD with unknown parameters, α, β, and θ defined previously.

The likelihood function of the BXPD using the pdf in equation (6) is given by:

$$L\left(\underline{X} \mid \alpha,\beta,\theta\right) = \left(\frac{2\alpha^2\beta\theta}{(1+\alpha)^2}\right)^n e^{\beta\sum_{i=1}^{n}x}\, e^{-\sum_{i=1}^{n}\left[\frac{\alpha\left(e^{\beta x_i}-1\right)}{1+\alpha}\right]^2}\prod_{i=1}^{n}\left\{\left(e^{\beta x_i}-1\right)\left(1-e^{-\left[\frac{\alpha\left(e^{\beta x_i}-1\right)}{1+\alpha}\right]^2}\right)^{\theta-1}\right\} \quad (34)$$

Let the natural logarithm of the likelihood function be, ℓ, therefore, taking the natural logarithm of the function equation (34) above gives:

$$\ell = n\log 2 + 2n\log\alpha + n\log\beta + n\log\theta - 2n\log(1+\alpha) + \beta\sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\left[\frac{\alpha\left(e^{\beta x_i}-1\right)}{1+\alpha}\right]^2$$
$$+\sum_{i=1}^{n}\log\left(e^{\beta x_i}-1\right) + (\theta-1)\sum_{i=1}^{n}\log\left(1-e^{-\left[\frac{\alpha\left(e^{\beta x_i}-1\right)}{1+\alpha}\right]^2}\right) \quad (35)$$

Differentiating ℓ partially with respect to α, β, and θ respectively gives the following results:

$$\frac{\partial \ell}{\partial \alpha}=\frac{2n}{\alpha}-\frac{2n}{1+\alpha}-2\alpha\sum_{i=1}^{n}\left[\frac{\left(e^{\beta x_i}-1\right)}{1+\alpha}\right]^2+2\alpha\left(\theta-1\right)\sum_{i=1}^{n}\left[\frac{\left(e^{\beta x_i}-1\right)^2\exp\left\{-\left[\frac{\alpha\left(e^{\beta x_i}-1\right)}{1+\alpha}\right]^2\right\}}{\left(1-\exp\left\{-\left[\frac{\alpha\left(e^{\beta x_i}-1\right)}{1+\alpha}\right]^2\right\}\right)\left(1+\alpha\right)^3}\right] \quad (36)$$

$$\frac{\partial \ell}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}x_i-2\alpha\sum_{i=1}^{n}\left[\frac{x_i e^{\beta x_i}\left(e^{\beta x_i}-1\right)}{\left(1+\alpha\right)^2}\right]+\sum_{i=1}^{n}\left[\frac{x_i e^{\beta x_i}}{\left(e^{\beta x_i}-1\right)}\right]$$
$$+2\alpha\left(\theta-1\right)\sum_{i=1}^{n}\left[\frac{x_i e^{\beta x_i}\left(e^{\beta x_i}-1\right)\exp\left\{-\left[\frac{\alpha\left(e^{\beta x_i}-1\right)}{1+\alpha}\right]^2\right\}}{\left(1-\exp\left\{-\left[\frac{\alpha\left(e^{\beta x_i}-1\right)}{1+\alpha}\right]^2\right\}\right)\left(1+\alpha\right)^2}\right] \quad (37)$$

$$\frac{\partial \ell}{\partial \theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-\left[\frac{\alpha\left(e^{\beta x_i}-1\right)}{1+\alpha}\right]^2}\right) \quad (38)$$

The maximum likelihood estimators of α, β, and θ can be obtained by equating (36), (37) and (38) to zero and solving for the solution of the non-linear system of equations. It is complicated to solve the above system of equations analytically and hence the Newton-Raphson's iteration method is applied using computer applications such as R or any other suitable software.

6. Applications to Real Life Datasets

This section evaluates the capability of the proposed Burr X-Perks distribution (BXPD) compared to other extensions of the Perks distribution such as the exponentiated Perks distribution (ExpPD) by Singh and Choudhary (2017), the Kumaraswamy-Perks distribution (KumPD) by Oguntunde *et al*. (2018), and the conventional Perks distribution (PD) by Perks (1932), using two real life datasets.

The two datasets are obtained from Ratan (2011) and have been used previously by Korkmaz and Erişoğlu (2014). For the first data set of 50 observations on burr (in the unit of millimeter), the hole diameter is 12 mm and the sheet thickness is 3.15 mm. For the second data set of 50 observations, hole diameter and sheet thickness are 9 mm and 2 mm respectively. Hole diameter readings are taken on jobs with respect to one hole, selected and fixed as per a predetermined orientation (Korkmaz and Erişoğlu, 2014). The datasets are as given below:

Data Set I: 0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16.

Data Set II: 0.06, 0.12, 0.14, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.22, 0.14, 0.06, 0.04, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.04, 0.14, 0.26, 0.18, 0.16.

The model selection is carried out based upon the value of the log-likelihood function evaluated at the MLEs (ll), Akaike Information Criterion, AIC, Consistent Akaike Information Criterion, CAIC, Bayesian Information Criterion, BIC, Hannan Quin Information Criterion, HQIC, Anderson-Darling (A*), Cramèr-Von Mises (W*) and Kolmogorov-smirnov (K-S) statistics. The details about the statistics A*, W* and K-S are discussed in Chen and Balakrishnan [24]. Meanwhile, the smaller these statistics are, the better the fit of the distribution is. The required computations are carried out using the R package "AdequacyModel" which is freely available from http://cran.r-project.org/web/packages/AdequacyModel/AdequacyModel.pdf.

The MLEs of the model parameters are computed together with some goodness-of-fit statistics for the fitted distributions and the results are presented the tables below for dataset I and II respectively.

Table 1: Maximum Likelihood Estimates based on Dataset I

Distribution	Parameter Estimates			
BXPD	$\hat{\alpha}=0.3767233$	$\hat{\beta}=8.541405$	$\hat{\theta}=0.8255771$	-
KumPD	$\hat{\alpha}=1.6375141$	$\hat{\beta}=3.802076$	$\hat{a}=2.1369847$	$\hat{b}=7.477812$
ExpPD	$\hat{\alpha}=2.5715090$	$\hat{\beta}=9.733208$	$\hat{\lambda}=2.1841032$	-
PD	$\hat{\alpha}=0.2847401$	$\hat{\beta}=9.950138$	-	-

Table 2: The statistics ll, AIC, CAIC, BIC and HQIC based on Dataset I

Distribution	*ll*	*AIC*	*CAIC*	*BIC*	*HQIC*
BXPD	-56.45661	-106.91323	-106.39149	-101.17716	-104.72890

KumPD	-55.20456	-102.40913	-101.52024	-94.76104	-99.49669
ExpPD	-51.69787	-97.39574	-96.87400	-91.65967	-95.21141
PD	-49.09733	-94.19466	-93.93934	-90.37061	-92.73844

Table 3: The A^*, W^*, K-S statistic and P-values Based on Dataset I

Distribution	A^*	W^*	*K-S*	*P-Value (K-S)*
BXPD	0.0727239	0.4359077	0.09815476	0.7211014
KumPD	0.124649	0.7623156	0.1368996	0.3058796
ExpPD	0.1816793	1.093489	0.1394642	0.2851593
PD	0.1110696	0.687046	0.1885332	0.05719129

The estimated PDFs and CDFs of the fitted distributions based on Dataset I are presented in the figure below.

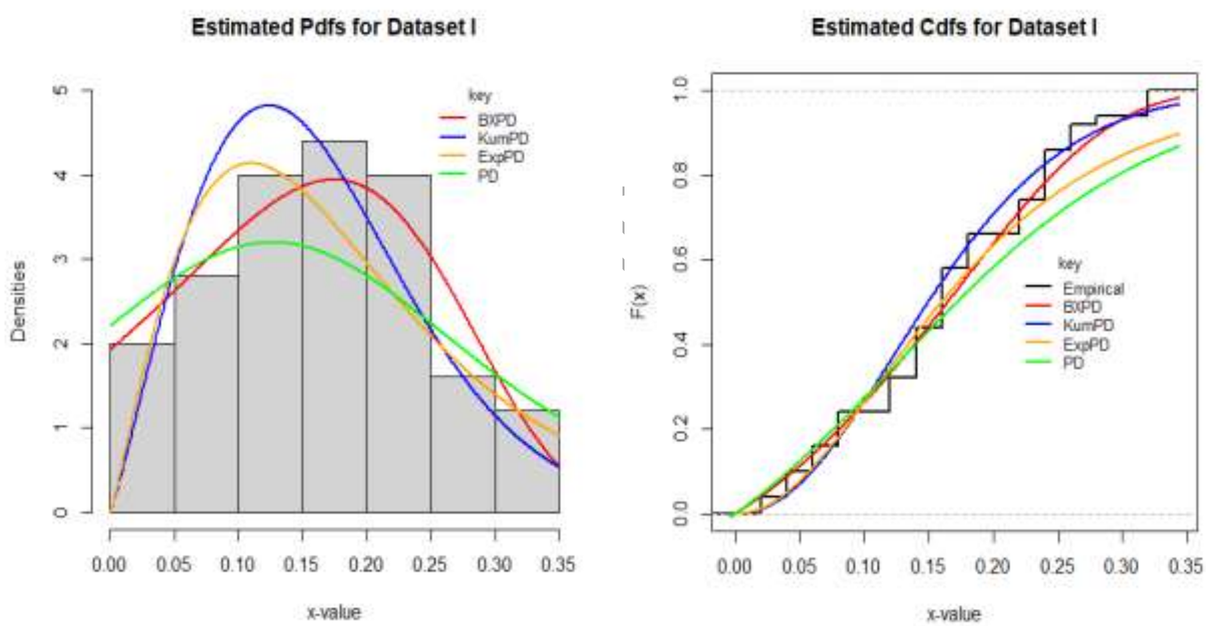

Figure 2: Estimated densities and CDFs of the fitted distributions based on Dataset I.

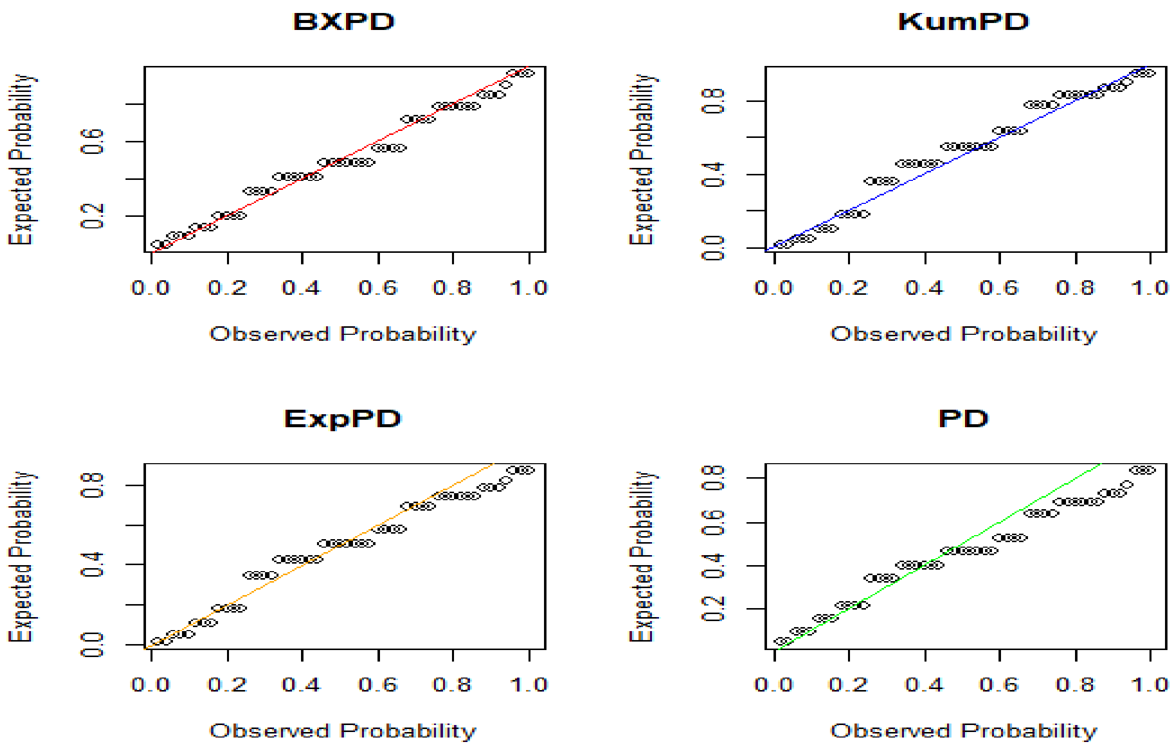

Figure 3: Probability plots for the fitted distributions based on Dataset I.

Table 4: Maximum Likelihood Estimates based on Dataset II

Distribution	Parameter Estimates			
BXPD	$\hat{\alpha} = 4.9646702$	$\hat{\beta} = 9.169217$	$\hat{\theta} = 0.3070174$	-
KumPD	$\hat{\alpha} = 1.6021568$	$\hat{\beta} = 4.197915$	$\hat{a} = 2.0588005$	$\hat{b} = 6.047402$
ExpPD	$\hat{\alpha} = 7.3175950$	$\hat{\beta} = 9.757824$	$\hat{\lambda} = 2.3340438$	-
PD	$\hat{\alpha} = 0.4238973$	$\hat{\beta} = 9.504486$	-	-

Table 5: The statistics ll, AIC, CAIC, BIC and HQIC Based on Dataset II

Distribution	*ll*	*AIC*	*CAIC*	*BIC*	*HQIC*
BXPD	-59.07203	-112.14406	-111.62233	-106.40800	-109.95974
KumPD	-56.45560	-104.91119	-104.02231	-97.26310	-101.99876
ExpPD	-52.90351	-99.80702	-99.28528	-94.07095	-97.62269
PD	-50.67092	-97.34184	-97.08652	-93.51779	-95.88562

Table 6: The A^*, W^*, K-S statistic and P-values Based on Dataset II

Distribution	A^*	W^*	*K-S*	*P-Value (K-S)*
BXPD	0.1094572	0.6774271	0.1667313	0.1240558
KumPD	0.2450171	1.377108	0.1583071	0.163081
ExpPD	0.3399511	1.871547	0.1554166	0.1785303
PD	0.2330336	1.319978	0.1966319	0.04186637

The estimated PDFs and CDFs of the fitted distributions based on Dataset II are presented in the figure below.

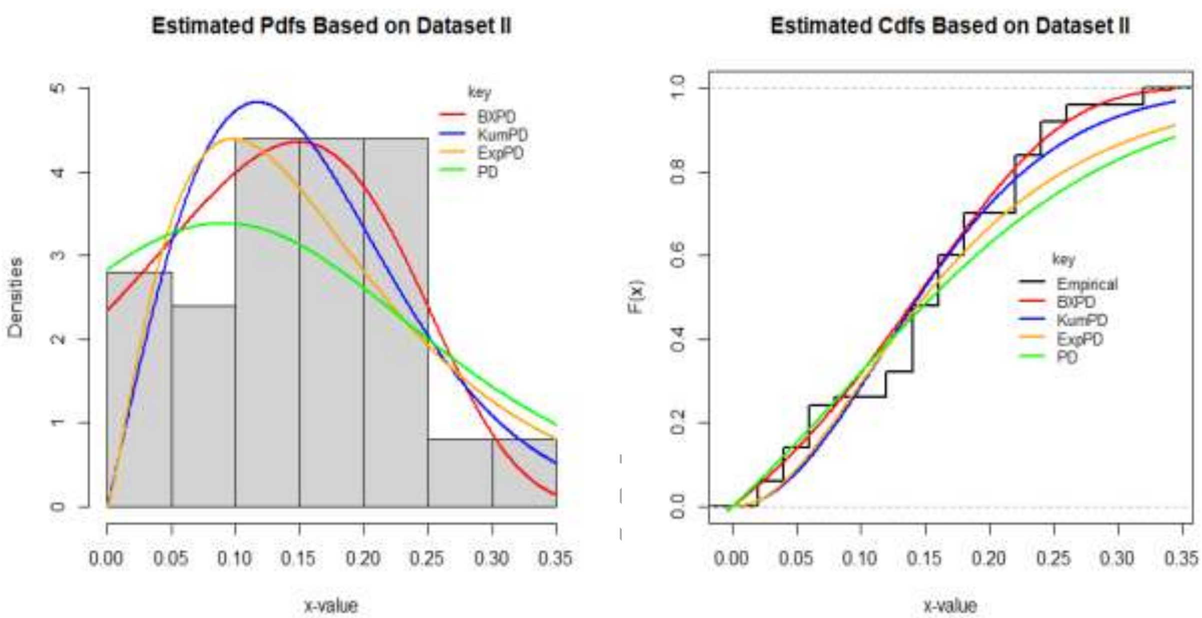

Figure 4: Estimated densities and CDFs of the fitted distributions based on Dataset II.

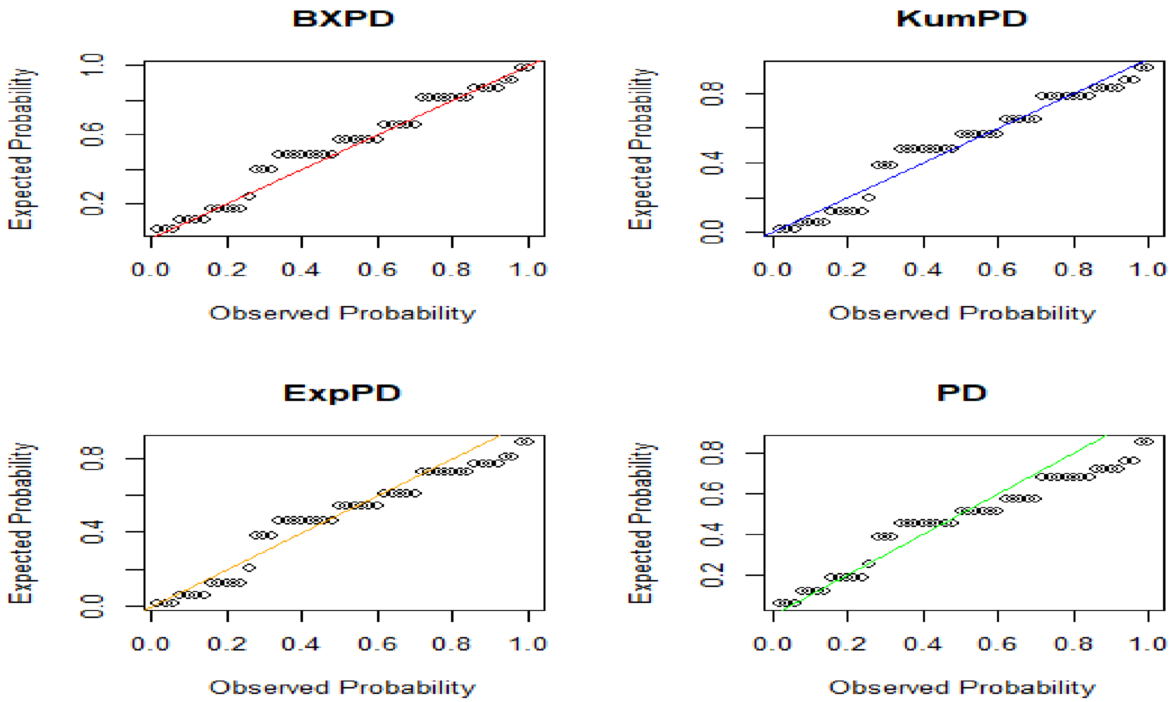

Figure 5: Probability plots for the fitted distributions based on Dataset II.

Tables 1 and 4 present the values of the Maximum Likelihood Estimates of the model parameters based on Dataset I and II respectively, while table 2 and 5 present the values of AIC, CAIC, BIC and HQIC for all the distributions fitted to Dataset I and II respectively, and the values of A*, W* and K-S for the fitted distributions based on Dataset I and II are provided in Tables 3 and 6 respectively. Also, the plots of the fitted densities and cumulative distribution functions of the fitted distributions for Dataset I and Dataset II are displayed in Figures 2 and 4 respectively and the probability plots of the fitted distributions for dataset I and dataset II are displayed in figures 3 and 5 respectively.

The values of AIC, CAIC, BIC and HQIC in Tables 2 and 5 for dataset I and dataset II respectively are smaller for the proposed Burr X-Perks distribution (BXPD) compared to the KumPD, ExpPD and the Perks distribution (PD). This result shows that the BXPD fits the two datasets better than the other three fitted distributions. The result is in line with the plots of the estimated densities and cumulative distribution functions of the fitted distributions displayed in Figures 2 and 4 as well as the probability plots in Figures 3 and 5 based on dataset I and dataset II respectively. Similarly, the values of A*, W* and K-S for the BXPD are on the average lower than the other three fitted distributions based on Dataset I and Dataset II as provided in Tables 3 and 6 respectively which is also a proof that the BXPD is better than the other distributions (exponentiated Perks distribution (ExpPD), the Kumaraswamy-Perks distribution (KumPD), and Perks distribution (PD))

The result of this study is in support of the fact that inducing additional shape parameter(s) into any standard probability distribution gives a compound distribution with a better fit to datasets than the standard one (Cordeiro *et al*., 2019; Reis *et al*., 2022; Bhat *et al*., 2023; Anzagra *et al*., 2022; Benchiha *et al*., 2023; Fayomi *et al*., 2023; Ieren *et al*., 2024).

7. Summary and Conclusion

In this paper a new compound probability distribution has been derived and studied. The new distribution known as "Burr X-Perks distribution (BXPD)". The paper derived and investigated some statistical properties of the new distribution. The paper also estimated the parameters of the BXPD using the method of maximum likelihood estimation. The plots of the probability density function and cumulative distribution function of the BXPD were also presented and discussed. The results from the curves show that the distribution is skewed with many shapes depending on the values of the parameters. The plot of the survival and hazard functions shows that the model can be used to model time-dependent events, where probability of survival decreases with time, while that of failure increases with time. Illustration of the performance of the Burr X-Perks distribution (BXPD) using two real life datasets revealed that the distribution fits the two real life datasets better than the other three distributions considered in this study.

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