ON BIVARIATE EXPONENTIAL DISTRIBUTION BASED ON ALI-MIKAILHAQ COPULA FUNCTION.

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Abstract

Copula function has become one of the most popular methods in constructing bivariate distributions. In this article a new bivariate generalized exponential distribution based on Ali-MikailHaqs copula function is introduced. Estimation of the parameters of the Ali-MikailHaq bivariate generalized exponential distribution was obtained via Bayesian method of estimation. An application of the proposed methodology was illustrated by fitting the distribution to a survival data set and compares its performance with other competing distribution. Based on the deviance information criteria (DIC) values, it is shown that, the Bivariate generalized exponential Ali-Mikha'il-Haq distribution is more efficient.

1. Introduction

Exponential distribution is a one-parameter continuous distribution that is usually used in measuring the amount of time for some specific event(s) to occur. The distribution is well known due to the constant hazard rate, memory less property and a decreasing probability density function it possesses (Usman, & Aliyu, 2022) and (Aliyu, & Usman, 2023). Hence, choosing the exponential distribution in reliability studies may be inappropriate since its hazard rate does not show monotone and/ or non-monotone failure rate behaviours (Tahir *et al*., 2018). To solve this problem, researchers have generalized the exponential distribution in order to add flexibility to the distribution. For instance, (Gupta & Kundu, 1999) generalizes the exponential distribution to the generalized exponential distribution, (Nadarajah & Haghighi, 2011) to the Nadarajah-Haghighi distribution. Other distributions such as Weibull, Gamma, Burr X, Burr XII, double exponential distributions also generalized the exponential distribution. The generalized exponential distribution serves as an alternative to the Weibull and gamma distributions. The probability density function (*pdf*), cumulative distribution function (*cdf*), reliability and hazard functions of the generalized exponential distribution are respectively given as:

$$f(t) = \vartheta\gamma exp(-\gamma t)\left(1 - exp(-\gamma t)\right)^{\vartheta} \tag{1}$$

$$F(t) = \left(1 - exp(-\gamma t)\right)^{\vartheta} \tag{2}$$

$$R(t) = 1 - \left(1 - exp(-\gamma t)\right)^{\vartheta} \tag{3}$$

and

$$h(t) = \frac{\vartheta\gamma exp(-\gamma t)\left(1-exp(-\gamma t)\right)^{\vartheta}}{1-\left(1-exp(-\gamma t)\right)^{\vartheta}} \tag{4}$$

where $\vartheta > 0$ is the shape parameter, $\gamma > 0$ is the scale parameter. The distribution reduces to the exponential distribution when the shape parameter (ϑ) takes the value one (1). This distribution can only model univariate lifetime.

However, the presence of bivariate data can be observed in different areas such as engineering, sciences and medicine. For example, the lifetimes of paired human organs, such as ears, eyes, kidneys, double recurrence of a certain disease and times to primary and secondary complications of a disease. Assume the lifetimes T_1 and T_2 be the times associated to the same device. Since in most bivariate lifetime, the time of one component may influence the time of the other component. That is, the bivariate times may presents dependence between the two times T_1 and T_2. To study the structure of this dependence, bivariate distributions through the use of frailty and copula functions are employed to study these dependences. The present paper, introduced bivariate generalized exponential distribution that could effectively model bivariate survival data where two lifetimes are observed for the same individual.

The rest of the paper is organized as follows: in section 2, we derive the survival function, the probability density function and cumulative distribution function of the Ali-Mikha'il-Haq bivariate generalized exponential distribution, some statistical properties of the distribution are derived in section 3, parameters of the distribution are estimated using the Bayesian estimation procedure in section 4. Application of the introduced methodology is given in section 5 and we finally conclude in section 6.

2. Bivariate Generalized Exponential Distribution

2.1 Ali-Mikhail-Haq Copula Function

Copulas are used to combine the joint distribution function of two or more univariate variables. A copula is said to be bivariate when it connects the joint distribution function of only two variables. Let $F_1(t_1)$ and $F_2(t_2)$ be the univariate *cdf* for the random variable T_1 and T_2 respectively, the joint *cdf* $F(u,\ v)$ is defined as:

$$F(t_1,\ t_2) = C_\lambda\big(F_1(t_1),\ F_2(t_2)\big) \tag{5}$$

where $0 < F_1(t_1),\ F_2(t_2) < 1$. Several types of copulas have been developed and studied. (Nelsen, 2006) and (Trivedi *et al.*, 2007) provides very thorough coverage of the various types of copulas. However, this study will use the Ali-Mikha'il-Haq copula function. The advantage of using this copula function, is that, it model both positive and negative dependence.

The Ali-Mikhail-Haq (AMH) copula function was first introduced by (Ali *et al.*, 1978) and was later discussed by (Kumar, 2010). The AMH copula function is defined as:

$$C_\mu\big(F_1(t_1),\ F_2(t_2)\big) = \frac{F_1(t_1)F_2(t_2)}{1-\mu S_1(t_1)S_2(t_2)} \tag{6}$$

where $S_1(t_1) = 1 - F_1(t_1)$, $S_2(t_2) = 1 - F_2(t_2)$ and μ is the dependence parameter and it lies in the interval $[-1, 1)$. Hence, the AMH copula function measure both positive and negative dependence. It reduces to the product copula when the dependence parameter takes the value zero.

2.2 The Model

The distribution function of the bivariate generalized exponential distribution based on *AMH* copula function is defined as:

$$F(t_1, t_2) = \frac{\big(1-exp(-\gamma_1 t_1)\big)^{\vartheta_1}\big(1-exp(-\gamma_2 t_2)\big)^{\vartheta_2}}{1-\lambda\Big[1-\big(1-exp(-\gamma_1 t_1)\big)^{\vartheta_1}\Big]\Big[1-\big(1-exp(-\gamma_2 t_2)\big)^{\vartheta_2}\Big]} \tag{7}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are scale parameters, $\vartheta_1 > 0$and $\vartheta_2 > 0$ are shape parameters and $\lambda \in (-1,\ 1)$ is a dependent parameter. The joint reliability function for the Ali-Mikhai-Haq generalized exponential (*AMHBGE*) distribution is given as:

$$R(t_1, t_2) = \frac{\Big[1-\big(1-exp(-\gamma_1 t_1)\big)^{\vartheta_1}\Big]\Big[1-\big(1-exp(-\gamma_2 t_2)\big)^{\vartheta_2}\Big]}{1-\lambda\big(1-exp(-\gamma_1 t_1)\big)^{\vartheta_1}\big(1-exp(-\gamma_2 t_2)\big)^{\vartheta_2}} \tag{8}$$

The joint cdf in equation (7) reduces to the joint cdf of the Ali-Mikhail-Haq exponential distribution when the shape parameters $\vartheta_1$and ϑ_2 takes the value one. That is, when $\vartheta_1 = \vartheta_2 = 1$.

3. Statistical properties of the *AMHBGE*

In this section, we derive some of the statistical properties for the *AMHBGE* distribution.

3.1 Partial derivatives

The first partial derivatives $\frac{\partial F(t_1,t_2)}{\partial t_1}$ and $\frac{\partial F(t_1,t_2)}{\partial t_2}$ for the *AMHBGE* distribution are obtain as follows:

$$\frac{\partial F(t_1,t_2)}{\partial t_1}=\frac{\vartheta_1\gamma_1 exp(-\gamma_1 t_1)\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\left\{1-\lambda\left[1-\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\right]\right\}}{\left[1-\lambda\left[1-\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\right]\left[1-\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\right]\right]^2} \tag{9}$$

and

$$\frac{\partial F(t_1,t_2)}{\partial t_2}=\frac{\vartheta_2\gamma_2 exp(-\gamma_2 t_2)\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\left\{1-\lambda\left[1-\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\right]\right\}}{\left[1-\lambda\left[1-\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\right]\left[1-\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\right]\right]^2} \tag{10}$$

The second partial derivative $\frac{\partial^2 F(t_1,t_2)}{\partial t_1 \partial t_2}$ for the *AMHBGE* distribution is given as:

$$\frac{\partial^2 F(t_1,t_2)}{\partial t_1\partial t_2}=\frac{1-\lambda+2\lambda\frac{\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}}{1-\lambda\left[1-\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\right]\left[1-\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\right]}}{\left[1-\lambda\left[1-\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\right]\left[1-\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\right]\right]^2} \tag{11}$$

Hence, the second partial derivative $\frac{\partial^2 F(t_1,t_2)}{\partial t_1 \partial t_2}$ is the joint *pdf* for the *AMHBGE* distribution. That is:

$$f(t_1,t_2)=\frac{1-\lambda+2\lambda\frac{\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}}{1-\lambda\left[1-\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\right]\left[1-\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\right]}}{\left[1-\lambda\left[1-\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\right]\left[1-\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\right]\right]^2} \tag{12}$$

3.2 Join hazard rate function

Basu (1971) defined the bivariate failure rate function as:

$$h(t_1,t_2)=\frac{f(t_1,t_2)}{R(t_1,t_2)} \tag{13}$$

Hence, the hazard rate function for the *AMHBGE* distribution is given as:

$$h(t_1,t_2)=\frac{\dfrac{1-\lambda+2\lambda\frac{\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}}{1-\lambda\left[1-\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\right]\left[1-\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\right]}}{\left[1-\lambda\left[1-\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\right]\left[1-\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\right]\right]^2}}{\dfrac{\left[1-\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\right]\left[1-\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}\right]}{1-\lambda\left(1-exp(-\gamma_1 t_1)\right)^{\vartheta_1}\left(1-exp(-\gamma_2 t_2)\right)^{\vartheta_2}}} \tag{14}$$

3.3 Marginal distribution function

Let $(T_1,T_2)\sim AMHBGE(\vartheta_1,\vartheta_2,\ \gamma_1,\ \gamma_2,\ \lambda)$, $T_1\sim GE(\vartheta_1,,\ \gamma_1)$ and $T_2\sim GE(\vartheta_2,,\ \gamma_2)$. The marginal distribution function of T_1 is obtained by evaluating:

$$f(t_1) = \int_0^\infty \frac{1-\lambda+2\lambda\frac{(1-exp(-\gamma_1 t_1))^{\vartheta_1}(1-exp(-\gamma_2 t_2))^{\vartheta_2}}{1-\lambda\left[1-(1-exp(-\gamma_1 t_1))^{\vartheta_1}\right]\left[1-(1-exp(-\gamma_2 t_2))^{\vartheta_2}\right]}}{\left[1-\lambda\left[1-(1-exp(-\gamma_1 t_1))^{\vartheta_1}\right]\left[1-(1-exp(-\gamma_2 t_2))^{\vartheta_2}\right]\right]^2} dt_2 \qquad (15)$$

and the marginal distribution function of T_2 is obtained by evaluating:

$$f(t_2) = \int_0^\infty \frac{1-\lambda+2\lambda\frac{(1-exp(-\gamma_1 t_1))^{\vartheta_1}(1-exp(-\gamma_2 t_2))^{\vartheta_2}}{1-\lambda\left[1-(1-exp(-\gamma_1 t_1))^{\vartheta_1}\right]\left[1-(1-exp(-\gamma_2 t_2))^{\vartheta_2}\right]}}{\left[1-\lambda\left[1-(1-exp(-\gamma_1 t_1))^{\vartheta_1}\right]\left[1-(1-exp(-\gamma_2 t_2))^{\vartheta_2}\right]\right]^2} dt_1 \qquad (16)$$

4. Estimation procedure

Bayesian estimation technique is a method that combines prior information with new information that is available to form the basis for statistical analysis. That is, Bayesian technique combines prior information with new information to come up with posterior distribution. To find the estimates of the model using Bayesian method, Let (T_{11}, T_{21}), (T_{12}, T_{22}), … (T_{1n}, T_{2n}) be bivariate random sample of size n from the *AMHBGE* distribution. Let $\Psi = (\varphi_1, \varphi_2, \gamma_1, \gamma_2, \lambda)'$ be the vector of parameters. Then, the likelihood function $L(\Psi)$ when the lifetimes (T_{11}, T_{21}), (T_{12}, T_{22}), … (T_{1n}, T_{2n}) is assumed to be non-censored can be expressed as:

$$L(\Psi) = \prod_{i=1}^{n} f(t_{1i}, t_{2i}) \qquad (17)$$

substituting equation (16) and taking natural logarithm give the log-likelihood function.

If the lifetimes T_1 or T_2 or both T_1 and T_2 may be right censored and assume censoring are independent. Then, each *ith* observation, $i = 1, 2, \cdots, n$ fall in one of the following groups:

i. G_1 : both t_{1i} and t_{2i} are uncensored observations.
ii. G_2 : t_{1i} is uncensored and t_{2i} is censored observation.
iii. G_3 : t_{1i} is censored and t_{2i} is uncensored observation.
iv. G_4 : both t_{1i} and t_{2i} are censored observations.

and the likelihood based on this condition is expressed as:

$$L(\Psi) = \prod_{i=1}^{n} \left(f(t_{1i}, t_{2i})\right)^{\delta_1\delta_2} \left(-\frac{\partial S(t_1,t_2)}{\partial t_1}\right)^{\delta_1(1-\delta_2)} \left(-\frac{\partial S(t_1,t_2)}{\partial t_2}\right)^{(1-\delta_1)\delta_2} \times \left(S(t_{1i}, t_{2i})\right)^{(1-\delta_1)(1-\delta_2)} \qquad (18)$$

substituting equations (8), (9), (10) and (12) into equation (18) gives the likelihood function for the *AMHBGE* distribution under the assumption of right censoring. It is important to note that, the likelihood function in equation (18) reduces to the likelihood function in equation (17) when the lifetimes T_1 and T_2 are not censored. That is, when $\delta_1 = \delta_2 = 1$.

However, under the Bayesian framework, the joint posterior distribution of the parameters in the model is obtained by combining the likelihood function and the joint prior distribution of the parameters. The likelihood function for the model parameters assuming right censoring is given in equation (18). We then assume the joint prior distribution for the model parameter to be $\Pi(\Psi)$, where $\Psi = (\varphi_1, \varphi_2, \gamma_1, \gamma_2, \lambda)'$. We then assume gamma prior for $\varphi_1, \varphi_2, \gamma_1$ and γ_2 while a uniform for λ is assumed since its parameter space is between ± 1. That is, assume:

$$\Pi(\varphi_k) \propto \varphi_k^{a_k - 1} e^{b_k \varphi_k} \qquad \varphi_k > 0 \qquad (19)$$

$$\Pi(\gamma_k) \propto \gamma_k^{c_k - 1} e^{d_k \gamma_k} \qquad \gamma_k > 0 \qquad (20)$$

for $k = 1, 2$ and

$$\Pi(\lambda) = \frac{1}{f-e} \tag{21}$$

where a_k, b_k, c_k, d_k, e and f are known hyper-parameters. Furthermore, we assumed independence between the prior distributions. Hence, the joint prior distribution $\Pi(\Psi)$ is given as:

$$\Pi(\Psi) \propto \varphi_1^{a_1-1} e^{b_1\varphi_1} \varphi_2^{a_2-1} e^{b_2\varphi_2} \gamma_1^{c_1-1} e^{d_1\gamma_1} \gamma_2^{c_2-1} e^{d_2\gamma_2} \frac{1}{f-e} \tag{22}$$

Hence, the joint posterior density is proportional to the product of the likelihood function in equation (18) and joint prior distribution in (22). Therefore, posterior summaries of interest are obtained by using Markov Chain Monte Carlo (MCMC) technique.

In our applications, 550,000 Gibbs samples for each model parameter is generated. In order to minimized the effect of initial values, the first 50,000 simulated samples were discarded as burn-in. Moreover, each 50th simulated sample was stored so as to avoid auto-correlation between successive samples. Hence, the Bayesian estimates for the parameters are based on 10,000 random samples. The medians of the respective posterior distributions are taken as the Bayesian estimate since some simulated distributions were quite skewed. Credible intervals were also determined for each model parameter from the 2.5th and 97.5th centiles of the posterior distribution of each model parameter. A generalization of the Akaike Information Criteria for the Bayesian analysis known as the Deviance information criteria (*DIC*) is used for discrimination.

5. Applications

In this section, two data sets: infections in kidney patients' data from (McGilchrist, & Aisbett, 1991) and Tobacco-stained-fingers data set from (John, *et al.*, 2015) are used in demonstrating the applicability of the *AMHBGE* distribution. The *AMHBGE* distribution is compared with its special cases: product bivariate generalized exponential (*PBGE*) and Ali-Mikha'il-Haq bivariate exponential distributions.

The kidney data showed the recurrence times to infection at point of insertion of catheter using portable dialysis equipment. Two recurrence times were recorded for each patient together with censoring indicator (Infection occurs =1 and censored=0). Assume T_1 and T_2 refers to first and second recurrence time respectively. The posterior summaries of the aforementioned distributions are given in Table 1.

Table 1: Posterior summaries for the Kidney data set

Model	Parameter	median	Sd	95% CrI	DIC
BGE AMH	γ_1	0.6932	0.1310	(0.4765, 0.9885)	259.6
	γ_2	0.1121	0.0411	(0.0533, 0.2122)	
	λ	0.7592	0.2604	(0.0466, 0.9908)	
	φ_1	0.4687	0.0727	(0.3407, 0.6244)	
	φ_2	0.6986	0.2454	(0.2942, 1.2600)	
BGE product	γ_1	0.5688	0.0741	(0.4397, 0.7300)	649.3
	γ_2	0.6239	0.0966	(0.4610, 0.8402)	
	φ_1	0.0479	0.0131	(0.0260, 0.0777)	
	φ_2	0.0580	0.0227	(0.0240, 0.1119)	
BEX	λ	0.9668	0.0579	(0.7899, 0.9990)	320.4
	φ_1	0.4762	0.0570	(0.3772, 0.6040)	
	φ_2	2.4950	0.3116	(1.9420, 3.1600)	

The posterior summary statistics for the *AMHBGE* distribution compared to *PBGE* and AMHBE distributions are given in Table 1. The results showed that, the *AMHBGE* distribution is more efficient than the PBGE and AMHBE distributions, since it has the least DIC value. In addition, the distributions based on *AMH* copula function are more efficient in fitting the data set than the bivariate distribution based on product copula function.

The second data set used in demonstrating the applicability of the developed distribution is the tobacco-stained-fingers data set. This data is obtained from John *et al.*, (2015) and the data consist of a sample of 143 smokers screened between March 2006 and January 2010 in a 180-bed community hospital in La Chauxde-Fonds, Switzerland. Information on death and hospital admission of the patients are collected until June 2014. For more details on this data set see John *et al.*, (2015). The posterior summaries for this data set are given in Table 2.

Table 2: Posterior summaries for the Tobacco-stained-fingers data set

Model	parameter	median	sd	95% CrI	DIC
	γ_1	0.2133	0.1579	(0.0829, 0.6803)	171.7
	γ_2	0.1178	0.0882	(0.0290, 0.3619)	
BGE	λ	0.7750	0.4812	(-0.7836, 0.9929)	
	φ_1	0.0054	0.0026	(0.0017, 0.0115)	
	φ_2	0.01271	0.0080	(0.0023, 0.0330)	
	γ_1	0.7476	0.1551	(0.1096, 1.0577)	689.1
BGE Product	γ_2	1.0510	0.2463	(0.0196, 1.5077)	
	φ_1	0.0063	0.0016	(0.0036, 0.0577)	
	φ_2	0.0080	0.0020	(0.0059, 0.0577)	
	λ	-0.1871	0.3277	(-0.8310, 0.4332)	197.2
BEX	φ_1	0.0094	0.0027	(0.0051, 0.0154)	
	φ_2	0.0350	0.0097	(0.0196, 0.0577)	

Table 2 gives the posterior summary statistics for the *AMHBGE* distribution compared to *PBGE* and AMHBE distributions. The results also showed that, the *AMHBGE* distribution is more efficient than the PBGE and AMHBE distributions, since it has the least DIC value. furthermore, the distributions based on *AMH* copula function (*AMHBGE* and *AMHBE* distributions) are more efficient in fitting the data set than the bivariate distribution based on product copula function.

Conclusion

In this work, a bivariate generalized exponential distribution based on Ali-Mikha'il-Haq copula function is introduced. Model parameters are estimated via Bayesian method of estimation. Kidney and Tobacco-stained-fingers data sets are used in demonstrating the applicability of the distribution. The performance of the distribution is compared with that of two Bivariate distributions: *PBGE* and *AMHBE* distributions. Based on the deviance information criteria (DIC) values, it is shown that, the Bivariate generalized exponential Ali-Mikha'il-Haq distribution is more efficient in fitting the data sets.

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