An Extended Sine Exponential Distribution with Applications to Reliability Engineering

Sule Omeiza Bashiru

Department of Mathematics and Statistics, Confluence University of Science and Technology, Osara, Kogi State, Nigeria

Email: bash0140@gmail.com

Abstract

This paper presents a new extension of the sine exponential distribution. The main statistical properties of the proposed model are studied, including the probability density function, cumulative distribution function, reliability functions, moments, quantile function, and mean residual life function. The density function of the model is unimodal and has a positively skewed shape. Four different methods are applied to estimate the unknown parameters of the model. To assess the performance of the new distribution, two real datasets from the field of reliability engineering are analyzed. The results from the data applications show that the proposed model provides better fits than other classical distributions, including the Marshall–Olkin exponential, sine exponential, and exponential distributions. These results confirm that the model is flexible and suitable for analyzing lifetime data in engineering and reliability studies.

Keywords: Exponentiated sine exponential distribution; mean residual life function; parameter estimation; lifetime data; goodness-of-fit

1. Introduction

Probability distributions play a central role in statistics, as they provide tools for modeling random phenomena and describing uncertainty in data. Over the years, many classical distributions have been developed and widely applied in fields such as engineering, medicine, finance, and the natural sciences. While these standard models are useful, they are often limited in flexibility, especially when dealing with datasets that show different levels of skewness or variation in their failure rate patterns. This has motivated researchers to construct new probability distributions by extending or modifying existing distributions using different methods (Bashiru et al., 2025b).

One common approach is the use of generalized families of distributions. These families introduce additional parameters that enhance flexibility and can be applied to baseline models to generate new distributions. Examples in the literature include the Kumaraswamy-G family by Cordeiro and de Castro (2011), the Weibull-G family by Bourguignon and Cordeiro (2014), the odd exponentiated half-logistic-G family by Afify et al. (2017), the generalized Burr X-G family by Aldahlan (2020), the log-logistic tan generalized family by Zaidi et al. (2021), the Arctan-X family by Alkhairy et al. (2021), the tangent Topp-Leone family by Nanga et al. (2022), the sine type II Topp-Leone-G family by Isa et al. (2023), the Marshall–Olkin exponentiated half-logistic-G family by Gabanakgosi and Otlaadisa (2024), the Ristić–Balakrishnan exponentiated half-logistic-G family by Oluyede and Rannona (2025), the cosine Kumarswamy-G family by Ali et al. (2025a), the cosine inverse Lomax-G family by Bashiru et al. (2025a), and the exponential Arctan-G family by Mohammad and Gaire (2025).

The sine exponential distribution, a one-parameter model proposed by Isa et al. (2023), has been used in modeling reliability data due to its analytical tractability and ability to describe non-negative, right-skewed data. Likewise, the exponentiated-G family of distributions introduced by Gupta et al. (1998) has proven effective for extending baseline models, providing more adaptable forms for a variety of real-life applications (see, for example, Abdul-Moniem and Abdel-Hameed (2012), Bashiru et al. (2024), and Ali et al. (2025b)).

Motivated by the flexibility of the exponentiated family and the practical relevance of the sine exponential distribution, this paper proposes the exponentiated sine exponential distribution (ESED). The new model is constructed by compounding the exponentiated family with the sine exponential distribution, resulting in a flexible distribution particularly suited for right-skewed datasets. The mathematical properties of the ESED are studied in detail, including the probability density function, survival function, hazard function, moments, quantile function, and moment generating function. Four estimation methods are employed for parameter estimation, ensuring consistency and efficiency in inference.

To evaluate the practical value of the proposed model, the ESED is applied to two real datasets. Comparative results with competing models demonstrate that the ESED provides a superior fit, underscoring its usefulness in statistical modeling of real-world data.

2. Materials and Methods

2.1 Formulation of the exponentiated sine exponential distribution

The cumulative distribution function (CDF) and probability density function (PDF) of the exponentiated family of distributions are defined in equations (1) and (2):

$$F(x) = [G(x)]^{\beta} \tag{1}$$

$$f(x) = \beta g(x)[G(x)]^{\beta-1} \tag{2}$$

where $\beta > 0$ is a shape parameter, and $g(x)$ and $G(x)$ denote the PDF and CDF of a baseline distribution, respectively.

The CDF and PDF of the sine exponential distribution are given in equations (3) and (4):

$$G(x) = \sin\left(\frac{\pi}{2}\left(1 - e^{-\lambda x}\right)\right) \tag{3}$$

where $\lambda > 0$ is a scale parameter.

$$g(x) = \frac{\pi}{2}\lambda e^{-\lambda x}\cos\left(\frac{\pi}{2}\left(1-e^{-\lambda x}\right)\right) \quad (4)$$

By substituting equation (3) into equation (1), the CDF of the proposed Exponentiated Sine Exponential Distribution (ESED) is obtained as:

$$F(x) = \left[\sin\left(\frac{\pi}{2}\left(1-e^{-\lambda x}\right)\right)\right]^{\beta} \quad (5)$$

where $\lambda, \beta > 0$ and $x \geq 0$.

Similarly, the corresponding PDF is derived by substituting equations (3) and (4) into equation (2):

$$f(x) = \beta\frac{\pi}{2}\lambda e^{-\lambda x}\cos\left(\frac{\pi}{2}\left(1-e^{-\lambda x}\right)\right)\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\lambda x}\right)\right)\right]^{\beta-1} \quad (6)$$

Figure 1 presents the PDF of the ESED. The plot indicates that the proposed model is right-skewed, meaning that it has a long tail extending to the right with most observations concentrated on the left side of the distribution.

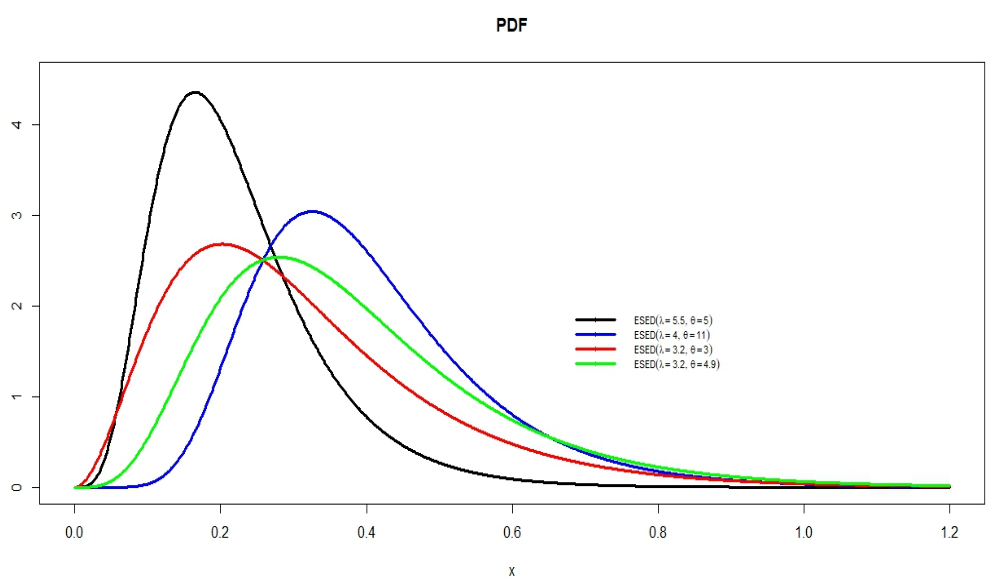

Figure 1. PDF plot of the ESED

The survival function (SF) represents the probability that a random variable exceeds a given time, and it is often used to study the likelihood of survival or reliability beyond that point. The hazard function (HF), on the other hand, gives the instantaneous rate of failure at a particular time, provided that the subject has survived up to that time. For the proposed distribution, the SF and HF are expressed as follows:

$$SF = 1 - F(x) = 1 - \left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\lambda x}\right) \right) \right]^{\beta}$$

$$HF = \frac{f(x)}{1 - F(x)} = \frac{\beta \frac{\pi}{2} \lambda e^{-\lambda x} \cos\left(\frac{\pi}{2}\left(1 - e^{-\lambda x}\right) \right) \left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\lambda x}\right) \right) \right]^{\beta - 1}}{1 - \left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\lambda x}\right) \right) \right]^{\beta}}$$

Figure 2 shows the HF plot of the ESED. The plot indicates that the proposed model exhibits a monotonically increasing hazard shape, which means that the risk of failure grows as time increases. This behavior is common in reliability and lifetime studies, where older units or systems are more likely to fail with age. Such a hazard structure makes the model suitable for applications in engineering, medical survival analysis, and other fields where failure risk increases over time.

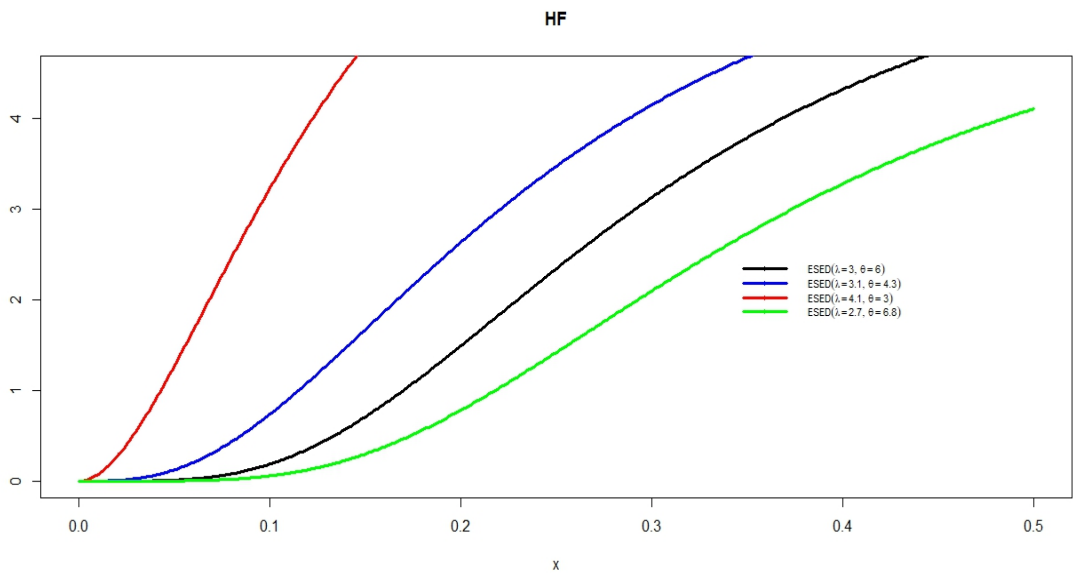

Figure 2 Hazard function plots of the ESED

2.2 Linear Representation of the PDF of the ESED

In this section, an expansion method is used to rewrite the PDF in equation (6) so that the properties of the proposed distribution can be derived more easily. The PDF in equation (6) can be expanded using Taylor's series as follows:

$$\cos\left(\frac{\pi}{2}\left(1 - e^{-\lambda x}\right) \right) = \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} \frac{\pi^{2i}}{2^{2i}} \left(1 - e^{-\lambda x}\right)^{2i}$$

$$\left\{\sin\left(\frac{\pi}{2}\left(1-e^{-\lambda x}\right)\right)\right\}^{\beta-1}=\sum_{j=0}^{\infty}\frac{(-1)^{j}}{(2j+1)!}\frac{\pi^{2j}}{2^{2j}}\left(1-e^{-\lambda x}\right)^{2j}$$

$$\left(1-e^{-\lambda x}\right)^{2j}=\sum_{k=0}^{\infty}(-1)^{k}\binom{2i+2j+1}{k}\left(e^{-\lambda x}\right)^{k}$$

Hence, the PDF in equation (6) can be expressed as:

$$f(x)=\sum_{i,j,k=0}^{\infty}\frac{(-1)^{i+j+k}}{(2i)!(2j+1)!}\frac{\pi^{2i+2j+1}}{2^{2i+2j+1}}\binom{2i+2j+1}{k}e^{-\lambda x(k+1)}$$

$$f(x)=\sum_{i,j,k=0}^{\infty}\beta\lambda\Phi_{i,j,k}e^{-\lambda x(k+1)} \qquad (7)$$

where, $\Phi_{i,j,k}=\frac{(-1)^{i+j+k}}{(2i)!(2j+1)!}\frac{\pi^{2i+2j+1}}{2^{2i+2j+1}}\binom{2i+2j+1}{k}$

2.3 Statistical Properties of the ESED

This subsection presents some statistical properties of the ESED.

2.3.1 Quantile Function

The quantile function, which is the inverse of the cumulative distribution function (CDF), is useful for generating random samples from the distribution. For the ESED, it is given by:

$$qf=-\frac{In\left(1-\frac{\sin^{-1}\left(u^{\frac{1}{\beta}}\right)}{\frac{\pi}{2}}\right)}{\lambda},0<u<1.$$

2.3.2 Moment

The $r-th$ moment of a random variable X is defined as

$$E(x^{r})=\int_{-\infty}^{\infty}x^{r}f(x)dx,$$

Moments are important because they provide information about key characteristics of a distribution, such as its mean, variance, skewness, and kurtosis. For the ESED, the $r-th$ moment can be derived from equation (7) as follows:

$$\mu_{r}=\sum_{i,j,k=0}^{\infty}\Phi_{i,j,k}\int_{0}^{\infty}x^{r}e^{-\lambda x(k+1)}dx=\int_{0}^{\infty}x^{r}e^{-\lambda x(k+1)}\,dx=\frac{r!}{[\lambda(k+1)]^{r+1}}.$$

2.3.3 Moment Generating Function

The moment generating function (MGF) is an important tool in probability theory, as it uniquely characterizes a distribution and can be used to obtain all moments of a random variable. It is generally defined as

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx,$$

Using the expression in equation (7), the MGF of the ESED is obtained as

$$M_x\left(t\right) = \sum_{i,j,k=0}^{\infty} \Phi_{i,j,k} \int_0^{\infty} e^{(t-\lambda(k+1))x}\, dx = \sum_{i,j,k=0}^{\infty} \Phi_{i,j,k} \frac{1}{\lambda(k+1)-t}.$$

2.3.4 Mean Residual Life Function

The mean residual life function (MRLF) is defined as

$$MRLF(x) = \frac{1}{S\left(m\right)} \int_m^{\infty} xf\left(x\right) dx - m,$$

where $S(m)$ denotes the survival function. It gives the expected remaining lifetime of a system or component that has already survived up to time m.

$$MRLF(x) = \frac{1}{S\left(m\right)} \sum_{i,j,k=0}^{\infty} \Phi_{i,j,k} \int_m^{\infty} x e^{-\lambda x(k+1)} dx - m = \frac{1}{S\left(x\right)} \sum_{i,j,k=0}^{\infty} \Phi_{i,j,k} \frac{e^{-((k+1)\lambda m)}((k+1)\lambda m+1)}{(k+1)^2 \lambda^2} - m.$$

2.4 Methods of Parameter Estimation

The methods used for the estimation of the parameters of the ESED are presented in this subsection.

2.4.1 Maximum Likelihood Estimation (MLE) Method

The maximum likelihood estimation (MLE) method is one of the most widely used techniques for estimating distributional parameters. It determines the parameter values that maximize the likelihood of observing the given data under the specified model. The method is based on the likelihood function, which expresses the probability of the observed data as a function of the parameters. By maximizing this function, the MLE provides estimates that best align the model with the data.

For the proposed ESED, the likelihood function corresponding to the PDF in equation (6) is given by:

$$\ell = n\log\left(\frac{\pi}{2}\right) + n\log\left(\beta\right) + n\log\left(\lambda\right) - \lambda\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log\cos\left(\frac{\pi}{2}\left(1-e^{-\lambda x_i}\right)\right) + (\beta-1)\sum_{i=1}^{n} \log\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\lambda x_i}\right)\right)\right]$$

Taking the partial derivative of the log-likelihood function with respect to β yields:

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log\left[\sin\left(\tfrac{\pi}{2}\left(1-e^{-\lambda x_i}\right)\right)\right] \tag{8}$$

Taking the partial derivative of the log-likelihood function with respect to α yields:

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\left[\tan\left(\tfrac{\pi}{2}\left(1-e^{-\lambda x_i}\right)\right) \tfrac{\pi}{2} x_i e^{-\lambda x_i} \right] + (\beta - 1)\sum_{i=1}^{n}\left[\cot\left(\tfrac{\pi}{2}\left(1-e^{-\lambda x_i}\right)\right) \tfrac{\pi}{2} x_i e^{-\lambda x_i} \right] \quad (9)$$

By setting equations (8) and (9) equal to zero, the maximum likelihood estimates of the ESED parameters are obtained. Since these equations are nonlinear and do not admit closed-form solutions, numerical methods are usually required to compute the parameter estimates. Common approaches include the Newton–Raphson method, the expectation–maximization (EM) algorithm, and other iterative optimization techniques.

2.4.2 Anderson Darling (AD) method

The Anderson–Darling (AD) method is based on the goodness-of-fit approach. It gives more weight to the tails of the distribution by comparing the empirical cumulative distribution function (ECDF) with the theoretical cumulative distribution function (CDF). Because of this, the method is useful in areas where tail behavior is important, such as reliability studies, risk analysis, and finance. For the ESED, the AD estimates of the parameters are obtained by minimizing the following function:

$$A(x_i) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log F(x_{i:n}) + \log S(x_{n-i-1:n})\right].$$

2.4.3 Cramer von Mises (CVM) method

The Cramér–von Mises (CVM) method estimates the parameters of a theoretical distribution by minimizing the CVM statistic, which measures the difference between the empirical cumulative distribution function (ECDF) and the theoretical CDF. This method provides a balanced assessment of the overall fit across the entire range of the data.

For the ESED, the CVM estimates of the parameters are obtained by minimizing the following function:

$$C(x_i) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^2.$$

2.4.4 Maximum Product of spacings (MPS) Method

The maximum product of spacings (MPS) method is an effective alternative to the maximum likelihood estimation (MLE) approach. Also called maximum product of spacings estimation (MPSE), it involves maximizing the product of the spacings between the ordered data points. This method is particularly useful for continuous distributions and can perform well even when MLE is difficult to apply.

For the proposed ESED, the MPS estimates of the parameters are obtained by maximizing the following function:

$$MPS = \frac{1}{n+1}\sum_{i=1}^{n+1}\log I_i(x_i)$$

where: $I_i(x_i) = F(x_{i:n}) - F(x_{i-1:n})$ and $F(x_{0:n})=0$ and $F(x_{n+1:n})=1$.

3. Results and Discussion

3.1 Simulation Studies

A simulation study was carried out to assess the performance of the four estimation methods, MLE, AD, CVM, and MPS, for the ESED with parameters λ = 1.5 and β = 1.7. Table 1 reports the estimates, biases, and mean squared errors (MSE) for each parameter across different sample sizes. The results show that, for all methods, the estimated values of λ and β converge toward the true parameter values as the sample size increases. In addition, both the bias and MSE decrease with larger samples, indicating improved accuracy and stability of the parameter estimates. Figures 3 to 6 illustrate the declining trend of bias and MSE for each parameter under the four estimation methods. These plots support the findings in Table 1, confirming that larger sample sizes lead to smaller bias and MSE.

Table 1. Parameter estimates with corresponding bias and MSE for different sample sizes.

Method	n	est $\hat{\lambda}$	est $\hat{\beta}$	Bias $\hat{\lambda}$	Mse $\hat{\beta}$	MSE $\hat{\lambda}$	MSE $\hat{\beta}$
MLE	20	1.6740	2.0600	0.3452	0.6044	0.2126	0.8172
	50	1.5580	1.8186	0.1932	0.3130	0.0619	0.1839
	100	1.5205	1.7566	0.1333	0.2183	0.0281	0.0822
	150	1.5135	1.7280	0.1080	0.1662	0.0183	0.0446
	300	1.5116	1.7177	0.0754	0.1164	0.0093	0.0224
	450	1.5064	1.7103	0.0607	0.0916	0.0056	0.0134
	600	1.5068	1.7117	0.0531	0.0789	0.0044	0.0101
	800	1.5050	1.7097	0.0451	0.0698	0.0033	0.0076
AD	20	1.5597	1.8495	0.3181	0.5130	0.1659	0.5304
	50	1.5262	1.7811	0.2017	0.3280	0.0673	0.2033
	100	1.5101	1.7454	0.1432	0.2271	0.0321	0.0851
	150	1.5076	1.7259	0.1129	0.1795	0.0204	0.0516
	300	1.5047	1.7059	0.0816	0.1226	0.0104	0.0241
	450	1.5025	1.7074	0.0668	0.1030	0.0070	0.0165
	600	1.4996	1.7024	0.0546	0.0840	0.0047	0.0110
	800	1.5000	1.7024	0.0504	0.0745	0.0039	0.0088
CVM	20	1.6687	2.1325	0.3922	0.7426	0.2839	1.2052
	50	1.5540	1.8371	0.2320	0.3798	0.0917	0.297
	100	1.5297	1.7626	0.1624	0.2531	0.0433	0.1081
	150	1.5225	1.7458	0.1310	0.2027	0.0274	0.0689
	300	1.5115	1.7263	0.0920	0.1460	0.0135	0.0347
	450	1.5103	1.7182	0.0719	0.1127	0.0080	0.0202
	600	1.5009	1.7095	0.0584	0.0962	0.0053	0.0142
	800	1.5019	1.7012	0.0540	0.0838	0.0046	0.0112
MPS	20	1.3910	1.5969	0.3111	0.4668	0.1494	0.3701
	50	1.4204	1.6099	0.1886	0.2936	0.0537	0.1321
	100	1.4539	1.6342	0.1388	0.1961	0.0295	0.0594
	150	1.4610	1.6434	0.1117	0.1740	0.0187	0.0457
	300	1.4735	1.6594	0.0778	0.1129	0.0094	0.0199
	450	1.4836	1.6764	0.0633	0.0958	0.0063	0.014
	600	1.4867	1.6820	0.0527	0.0821	0.0043	0.0106
	800	1.4905	1.6868	0.0471	0.0701	0.0035	0.0079

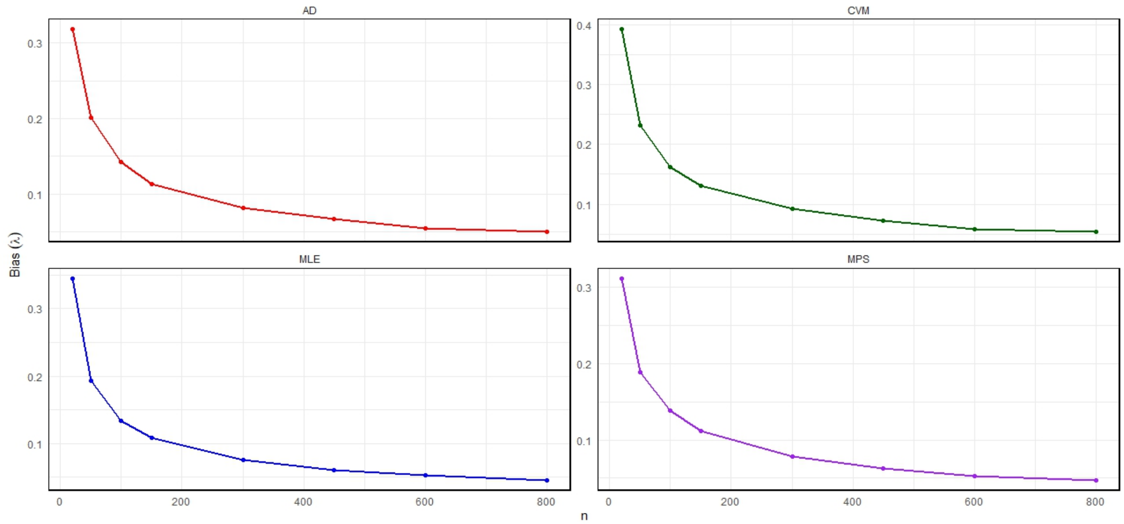

Figure 3. Bias of the estimator of parameter λ.

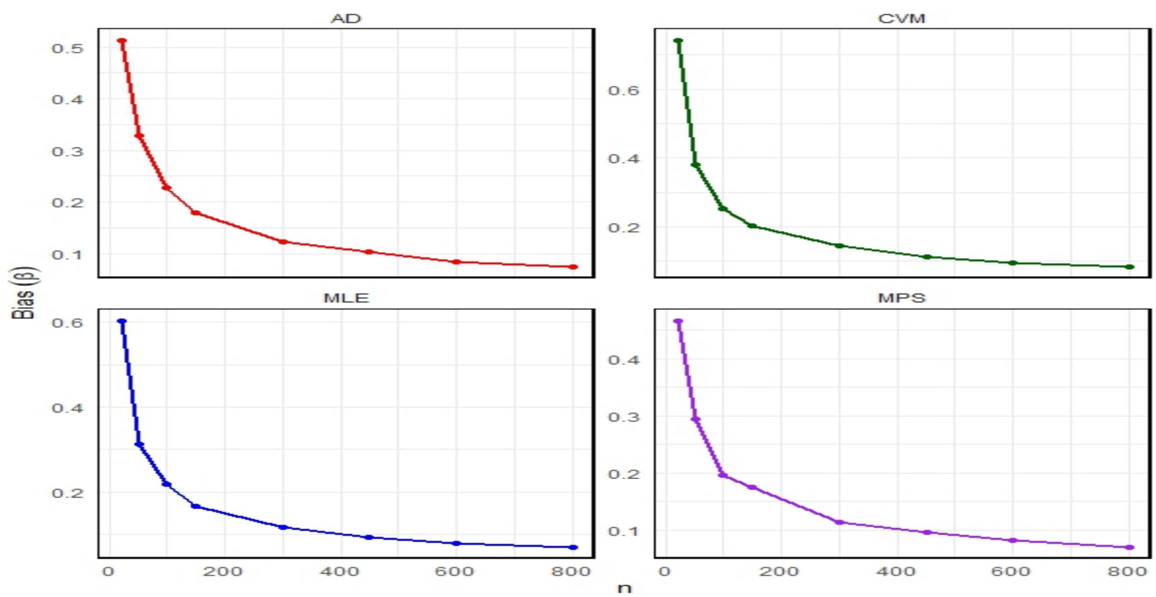

Figure 4. Bias of the estimator of parameter β.

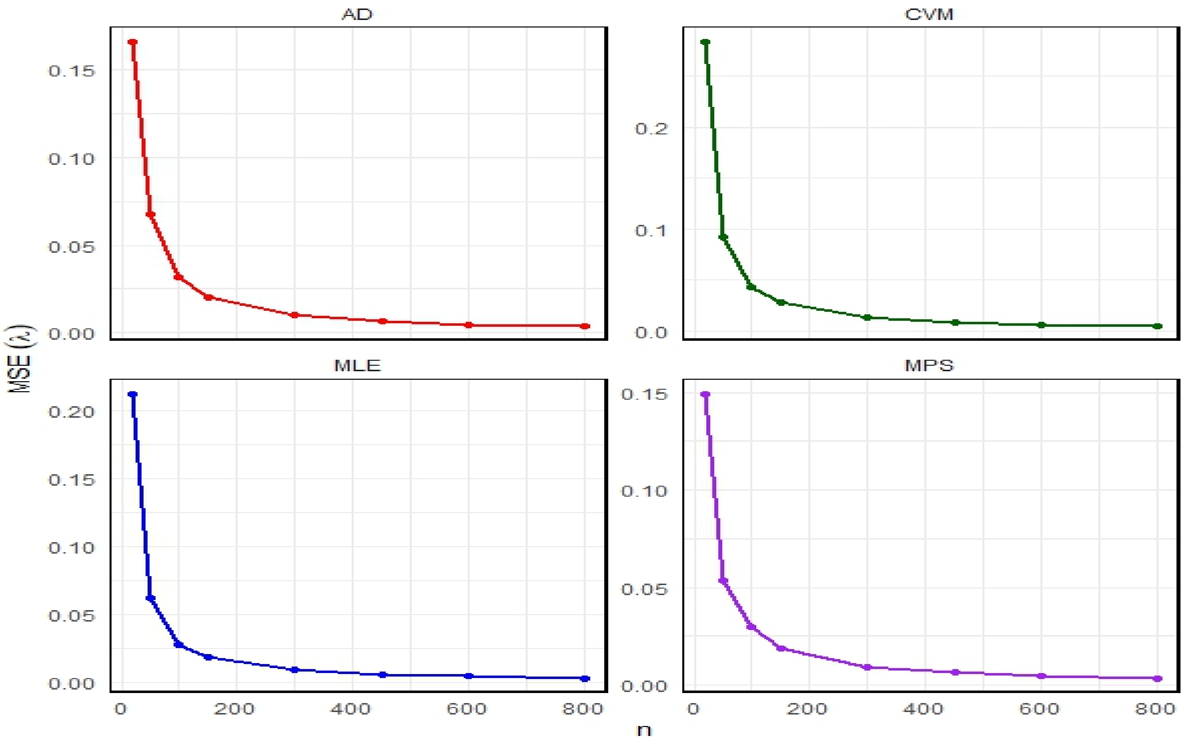

Figure 5. MSE of the estimator of parameter λ.

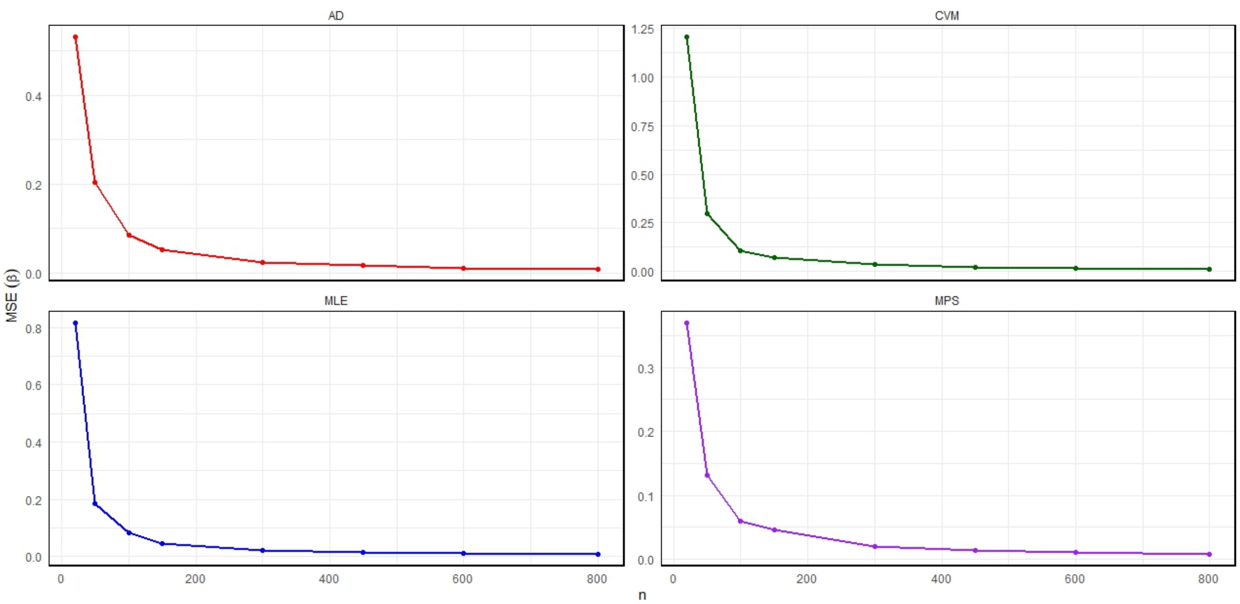

Figure 6. MSE of the estimator of parameter β.

3.2 Real Data Application

To evaluate the practical performance of the proposed ESE model, it was applied to two real-life right-skewed datasets. The ESED was compared with the Marshall–Olkin exponential distribution (MOED) as discussed by Salah (2016), sine exponential distribution (SED), and the exponential distribution (ED). The performance was assessed using negative log-likelihood (−LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), root mean square error (RMSE), and the Kolmogorov–Smirnov (KS) statistic with its associated p-value. The best model is the one with the lowest −LL, AIC, BIC, HQIC, RMSE, and KS statistic, together with the highest p-value.

The first dataset, reported by Lawless (1982), gives the number of million revolutions before failure for 23 ball bearings:

17.88, 28.92, 33.0, 41.52, 42.12, 45.6, 48.8, 51.84, 51.96, 54.12, 55.56, 67.8, 68.44, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 105.84, 127.92, 128.04, 173.4.

The descriptive statistics of this dataset are shown in Table 2. The results indicate right-skewness, with a skewness of 0.7972 and kurtosis of 3.1410. Table 3 presents the goodness-of-fit measures for the competing models. The ESED achieved the smallest values of −LL, AIC, BIC, HQIC, RMSE, and KS statistic, along with the highest p-value, showing that it provides the best fit among the models. The fitted PDF and CDF plots for the first dataset are given in Figure 7. These plots confirm that the ESED follows the data more closely than the other models.

Table 2. Descriptive Statistics of the first data

Min	Max	Mean	Median	Skewness	Kurtosis
17.88	173.40	73.85	67.80	0.7972	3.1410

Table 3. Goodness-of-fit statistics for the first dataset

MODELS	MLE	-LL	AIC	BIC	HQIC	RMSE	KS	P value
ESED	$\hat{\beta} = 4.3136$ $\hat{\lambda} = 0.0157$	113.8017	231.6035	233.8744	232.1746	0.0460	0.1110	0.9397
MOED	$\hat{\alpha} = 17.4204$ $\hat{\beta} = 0.0418$	114.9943	233.9886	236.2596	234.5598	0.0551	0.1321	0.8172
SED	$\hat{\lambda} = 0.0077$	120.9485	243.8969	245.0324	244.1825	0.1283	0.2877	0.0443
ED	$\hat{\lambda} = 0.0135$	121.9459	245.8918	247.0273	246.1773	0.1370	0.2998	0.0320

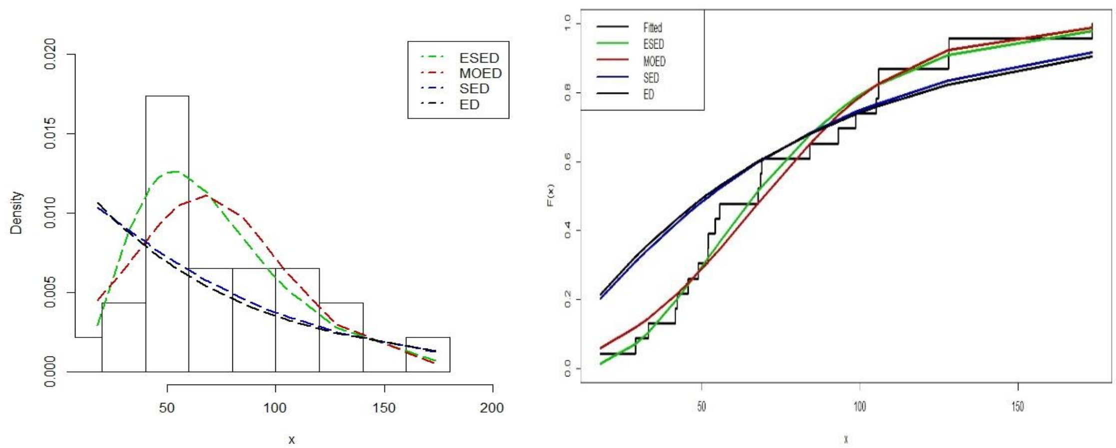

Figure 7. Fitted PDF and CDF of the ESED and competing models for the first dataset.

The second dataset, discussed by Murthy et al. (2004), gives the failure times of 24 mechanical components:
30.94, 18.51, 16.62, 51.56, 22.85, 22.38, 19.08, 49.56, 17.12, 10.67, 25.43, 10.24, 27.47, 14.70, 14.10, 29.93, 27.98, 36.02, 19.40, 14.97, 22.57, 12.26, 18.14, 18.84.
The descriptive statistics of this dataset are shown in Table 4. The data are right-skewed, with a skewness of 1.3454 and kurtosis of 4.3599. Table 5 presents the goodness-of-fit results for the competing models. Similar to the first dataset, the ESED produced the lowest values of −LL, AIC, BIC, HQIC, RMSE, and KS statistic, and the highest p-value, confirming that it provides the best fit to the data. The fitted PDF and CDF plots for the second dataset are displayed in Figure 8. These plots show that the ESED follows the observed data more closely than the other models.

Table 4. Descriptive Statistics of the second data

Min	Max	Mean	Median	Skewness	Kurtosis
10.24	51.56	22.97	19.24	1.3454	4.3599

Table 5. Goodness-of-fit statistics for the second dataset

MODELS	MLE	-LL	AIC	BIC	HQIC	RMSE	KS	P value
ESED	$\hat{\beta}=8.9283$ $\hat{\lambda}=0.0662$	86.1760	176.3521	178.7082	176.9771	0.0469	0.1223	0.8237
MOED	$\hat{\alpha}=36.6638$ $\hat{\beta}=0.1693$	88.9502	181.9004	184.2565	182.5255	0.0538	0.1294	0.7691
SED	$\hat{\lambda}=0.0250$	98.0465	198.0930	199.2711	198.4055	0.1640	0.3471	0.0043
ED	$\hat{\lambda}=0.0435$	99.2231	200.4463	201.6244	200.7588	0.1710	0.3598	0.0027

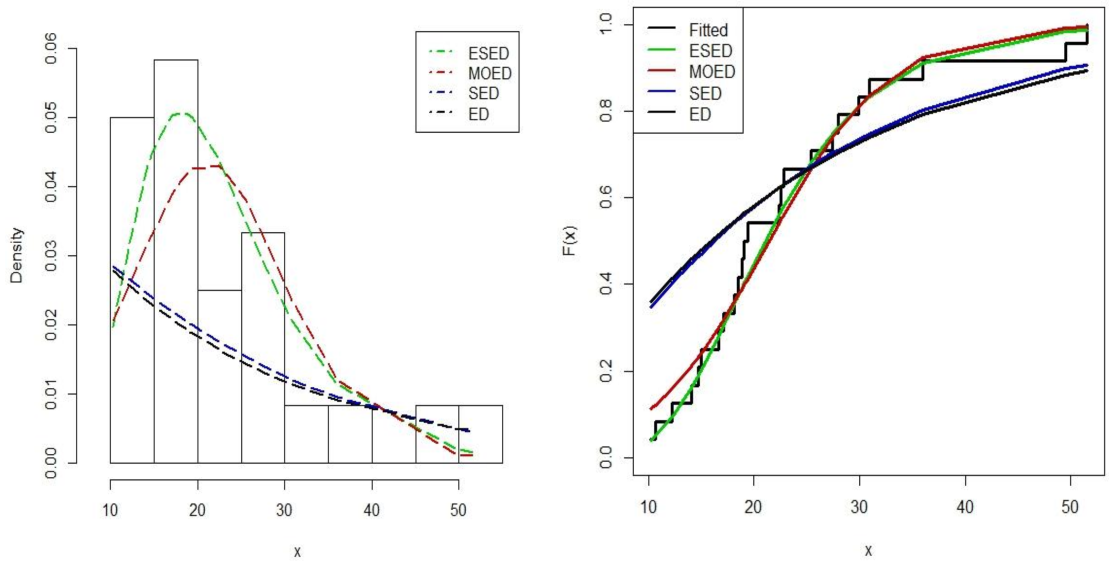

Figure 8. Fitted PDF and CDF of the ESED and competing models for the second dataset.

4. Conclusion

This study introduced the Exponentiated Sine Exponential Distribution (ESED), a new model obtained by compounding the exponentiated family of distributions with the sine exponential distribution. Important properties of the distribution were derived, including the hazard function, survival function, moments, quantile function, and moment generating function. The probability density function was shown to be unimodal and right-skewed. Four estimation methods were considered, and the simulation results indicated that all the estimators provide consistent estimates as sample size increases. The model was applied to two real datasets, and in both cases, the ESED gave the best fit compared to other competing models. These results show that the ESED is a useful tool for modeling right-skewed data. Further studies may explore the application of the ESED to censored and truncated data. Alternative estimation methods, such as Bayesian approaches, can be investigated. Another promising area is testing the model on larger and more varied datasets across different scientific fields to further assess its performance.

References

Abdul-Moniem, I. B., & Abdel-Hameed, H. F. (2012). On exponentiated Lomax distribution. *International Journal of Mathematical Archive, 3*(5), 1–7.

Afify, A. Z., Altun, E., Alizadeh, M., Ozel, G., & Hamedani, G. (2017). The odd exponentiated half-logistic-G family: Properties, characterizations and applications. *Chilean Journal of Statistics, 8*(2), 65–91.

Aldahlan, M. A. D., Khalil, M. G., & Afify, A. Z. (2020). A new generalized family of distributions for lifetime data. *Journal of Modern Applied Statistical Methods, 19*(1), eP2948.

Ali, I., Isa, A. M., Bashiru, S. O., & Chinedu, A. K. (2025a). Cosine Kumaraswamy family of distributions: Properties and applications to real-world datasets. *Sigma Journal of Engineering and Natural Sciences, 43*(6), 2186–2197.

Ali, I., Isa, A. M., Bashiru, S. O., & Itopa, I. I. (2025b). Enhancing the Cosine-Lomax distribution: A new exponentiated version with improved flexibility and real-world applications. *Iraqi Statisticians Journal*, *2*(2), 1–14.

Alkhairy, I., Nagy, M., Muse, A. H., & Hussam, E. (2021). The Arctan-X family of distributions: Properties, simulation, and applications to actuarial sciences. *Complexity, 2021,* 4689010.

Bashiru, S. O., Isa, A. M., Khalaf, A. A., Khaleel, M. A., Arum, K. C., & Anioke, C. L. (2025a). A hybrid cosine inverse Lomax-G family of distributions with applications in medical and engineering data. *Nigerian Journal of Technological Development, 22*(1), 261–278.

Bashiru, S. O., Kayid, M., Sayed, R. M., Balogun, O. S., Hammad, A. T., & Abd El-Raouf, M. M. (2025b). Transmuted inverse unit Teissier distribution: Properties, estimations and applications to medical and radiation sciences. *Journal of Radiation Research and Applied Sciences, 18*(1), 101208.

Bashiru, S. O., Khalaf A. A., Isah A. M. and Kaigama A. (2024). On Modeling of Biomedical Data with Exponentiated Gompertz Inverse Rayleigh Distribution. *Reliability: theory and applications.* 19(3): 460 – 475.

Bourguignon, M., Silva, R. B., & Cordeiro, G. M. (2014). The Weibull-G family of probability distributions. *Journal of Data Science, 12*(1), 53–68.

Cordeiro, G. M., & de Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation, 81*(7), 883–898.

Gabanakgosi, M., & Otlaadisa, M. (2024). A new family of generalized-G distributions with properties and applications. *Scientific African, 26,* e02344.

Gupta, R. C., Gupta, P. L., & Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives. *Communications in Statistics -Theory and Methods,* 27(4), 887–904.

Isa, A. M., Bashiru, S. O., Ali, B. A., Adepoju, A. A., & Itopa, I. I. (2022). Sine-exponential distribution: Its mathematical properties and application to real dataset. UMYU Scientifica, 1(1), 127–131

Isa, A. M., Doguwa, S. I., Alhaji, B. B., & Dikko, H. G. (2023). Sine type II Topp-Leone G family of probability distribution: Mathematical properties and application. *Arid Zone Journal of Basic and Applied Research, 2*(4), 124–138.

Lawless, J. F. (1982). *Statistical models and methods for lifetime data*. New York: John Wiley & Sons.

Mohammad, S., & Gaire, A. K. (2025). Exponential Arctan-G family of distributions with properties and applications. *Journal of Probability and Statistics, 2025,* 3643496.

Murthy, D. N. P., Xie, M., & Jiang, R. (2004). *Weibull models*. Hoboken, NJ: John Wiley & Sons.

Nanga, S., Nasiru, S., & Dioggban, J. (2022). Tangent Topp-Leone family of distributions. *Scientific African, 17,* e01363.

Oluyede, B., & Rannona, K. (2025). The Ristić-Balakrishnan-exponentiated half logistic-G family of distributions with applications. *International Journal of Mathematics in Operational Research, 31*(4), 518–543.

Salah, M. M. (2016). Parameter estimation of the Marshall–Olkin exponential distribution under type-II hybrid censoring schemes and its applications. *Journal of Statistics Applications & Probability, 5*(3), 377–384.

Zaidi, S. M., AL Sobhi, M. M., El-Morshedy, M., & Afify, A. Z. (2021). A new generalized family of distributions: Properties and applications. *AIMS Mathematics, 6*(1), 456–476. https://doi.org/10.3934/math.2021028